The Gamma/Exponential Markov Process 10/2/88

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ABSTRACT

The exponential autoregressive (EAR) time series process of Gaver and Lewis, EAR(1), is generalized to continuous time. It is shown that observation of the continuous time process at equally spaced time intervals recovers the EAR(1). We observe that the continuous time process is somewhat analogous to the Ornstein-Uhlenbeck process. While the latter is the stationary, time-reversible, Markov process with Gaussian marginal distribution produced by filtered Gaussian white noise, the continuous EAR(1) process is the stationary, irreversible, Markov process with exponential marginal distribution produced by filtered shot noise impulses having exponentially distributed amplitudes. The "defective" distribution of the EAR(1) innovations (non-zero probability of zero values) arises naturally from the integrated Campbell process. When the filter decay parameter does not exactly match the rate of shot impulses a gamma process results. The joint (multi-time) characteristic function of the discrete EAR(1) and the characteristic functional of the continuous process are derived and some applications are discussed. The characteristic functional for the corresponding second order continuous process (driven by the same input noise) is derived. Although it is also irreversible and possibly non-negative (for over-damping) it is not a gamma process.

I. INTRODUCTION

In seeking general models of stationary time series for applications in which the common assumption of Gaussianity (hence, time reversibility[1]) is not appropriate[2,3,4], Gaver and Lewis[5] modified the usual first order autoregressive model, AR(1),

$$x_n = \alpha x_{n-1} + \xi_n . \qquad (1)$$

They determined the properties of the innovation, ξ_n, such that the x's would have a marginal exponential distribution. Solving the corresponding equation for the characteristic functions of x and ξ, they found that ξ must have an exponential distribution with probability $1-\alpha$ and be zero with probability α. They termed this a "defective" distribution (by defective they mean not the usual definition[6] but that α could be determined from two successive observations, if the innovation is zero). If the innovation takes the value zero with probability $\alpha' \neq \alpha$ then the marginal distribution of the x's is Gamma.

The joint multi-time characteristic function of the discrete EAR(1) process is derived in Appendix A and applied to determining the sampling distribution of the periodogram. In Section II the EAR(1) process is generalized to continuous time. The input noise is found to be shot noise impulses with exponentially distributed amplitude. The "defective" distribution of the EAR(1) innovations (non-zero probability of zero values) is found to arise naturally from the integrated Campbell process. It is shown that observation of the continuous time process at equally spaced time intervals recovers the EAR(1).

The characteristic functional of the continuous process is derived and certain applications are investigated in Section III. In Appendix B the characteristic functional is found for the second order system driven by the same noise, showing that it is not a Gamma process.

We observe that the continuous time process is somewhat analogous to the Ornstein-Uhlenbeck process. While the latter is the stationary (time reversible) Markov process with Gaussian marginal distribution produced by filtered Gaussian white noise, the continuous EAR(1) process is the stationary (irreversible) Markov process with exponential marginal distribution produced by filtered shot noise impulses having exponentially distributed amplitudes. When the filter decay parameter does not exactly match the (possibly non-homogeneous) rate of shot impulses a gamma process results[8]. In fact, this is the first example of irreversibility in reference [1]. It provides a two-parameter family of non-negative, non-Gaussian, irreversible, Markov processes with which to model naturally occurring time series.

II. The Continuous Time Process

Let the continuous time parameter, t, be related to the discrete time index, n, by $t = n \cdot h$, where h is the time interval between observations of the time series. Note that it is reasonable for the inter-observation decay constant, α, to depend on the time between observations, h. Subtract αx_{n-1} from both sides of Eq. (1), and add and subtract x_{n-1} from the lhs, then divide by h to get,

$$(x_n - x_{n-1})/h + (1 - \alpha)/h \cdot x_{n-1} = \xi_n / h. \qquad (2)$$

Take the limit as h --> 0 and define, (clearly α --> 1),

$$\gamma(t) = \lim_{h \to 0} (1 - \alpha)/h,$$

where we have generalized to a non-homogeneous process. Although the scale of the continuous time innovation becomes infinite, the probability that it is non-zero goes to zero as h, i.e., $(1 - \alpha)$ --> $\gamma(t)\, h$. Hence, the innovation becomes a non-homogeneous, exponentially distributed (in magnitude), series of shot impulses (Campbell process). The continuous limit of Eq. (2) can be written as the stochastic differential equation,

$$d/dt\, x + \gamma(t)\, x = g(t), \qquad (3)$$

where g(t) is a shot noise process consisting of random delta function impulses multiplied by a r.v. (z_i) with exponential distribution,

$$g(t) = \sum z_i\, \delta(t - t_i). \qquad (4)$$

We define this more explicitly in Section III. In fact, it is really more proper to write Eq. (3) in the form,

$$dx(t) + \gamma(t)\,x(t) = dY(t) \qquad (5)$$

where Y(t) is a point process whose independent increments correspond to those of a non-homogeneous counting process (of rate $\gamma(t)$) times an exponentially distributed r.v. with parameter λ. Solving Eq. (3) is straightforward, yielding,

$$x(t+h) = \exp[-\int_t^{t+h} \gamma\, dt'\,] \cdot x(t) + \int_t^{t+h} \exp[\int_{t'}^{t+h} \gamma(t'')\, dt''\,]\, g(t')\, dt'\,. \qquad (6)$$

If we define $\alpha = \exp[-\int_t^{t+h} \gamma\, dt'\,]$, then the first term on the rhs is just αx_n, the second term is the innovation, while the lhs is x_{n+1} (cf. Eq.(1)). The probability of no arrivals in a non-homogeneous Poisson process is also given by α, as defined here, hence, we see that the second term is zero with probability α. To complete the recovery of the discrete EAR(1) process we need only show that when the innovation in Eq. (6) is non-zero it has an exponential distribution with parameter λ. If h is small enough so that the probability of more than one arrival between t, t+h, is negligible, this follows easily. For larger values of h the innovation (filtered shot impulses) will still have the required distribution. We defer the proof to the next Section where it follows easily using the characteristic functional derived there.

Given the above, we can write down almost immediately certain fundamental results for this process. For simplicity we now assume constant γ, the more general situation is easily recovered. The conditional pdf for $x(t)$, given $x(0) = x_0$, is,

$$f(x, t|x_0) = e^{-\gamma t}\delta(x - x_0 e^{-\gamma t}) + (1 - e^{-\gamma t})\lambda e^{-\lambda(x - x_0 \exp[-\gamma t])}. \qquad (7)$$

In other words, with probability $\alpha = e^{-\gamma t}$ the noise source remained zero and x decayed exponentially from its initial value, and with probability $(1-\alpha)$ an innovation with exponential distribution was added to this decaying value. Multiplying Eq. (7) by $\lambda e^{-\lambda x_0}$ yields the steady state joint pdf, $f(x,t; x_0)$. A straightforward calculation using this yields,

$$\mathrm{Cov}[x(t), x(0)] = 1/\lambda^2 e^{-\gamma t}, \qquad (8)$$

and, for the correlation function,

$$\rho(\tau) = e^{-\gamma|\tau|}, \qquad (9)$$

as expected for an exponentially distributed Markov process. This process is also discussed in reference [8].

<u>III. The Characteristic Functional</u>

We now derive the characteristic functional[7] for the continuous process. First, the shot-noise source, $g(t)$, is defined more explicitly over $[-T, T]$ as,

$$g(t) = \lim_{T \to \infty} \sum_{i=1}^{m} z_i \, \delta(t - t_i), \qquad (10)$$

where m has a non-homogeneous Poisson distribution, p_m, with mean,

$$\bar{n} = \int_{-T}^{T} \gamma(t)dt, \tag{11}$$

the t_i have pdf given by $\gamma(t)/\bar{n}$ (as is appropriate for a non-homogeneous Poisson process), and z_i have an exponential distribution, parameter λ. The characteristic functional for g(t) is,

$$C_g[\eta(\cdot)] = \lim_{T\to\infty} E\{ \exp[i\int_{-T}^{T}\eta(t)\, g(t)\, dt] \}, \tag{12}$$

or,

$$= \lim_{T\to\infty} \sum_{m=0}^{\infty} p_m \{ \prod_{j=1}^{m} [\int\int dt_j\, \gamma(t_j)/\bar{n}\; dz_j\, \lambda e^{-\lambda z_j} e^{i z_j \eta(t_j)}] \},$$

where the integration over t_j is -T -> T the integration over z_j is 0->∞. We have made use of the Dirac delta function to perform the integration over t. The product over j now reduces simply to the expression in the square brackets raised to the m^{th} power because of the independence of each term in the shot noise. Performing the average over m (yielding the standard result for the generating function of a Poisson distribution) and the integration over z_j (yielding the characteristic function for the exponential distribution) and taking the limit T --> ∞, results in,

$$C_g[\eta(\cdot)] = \exp \{ i \int_{-\infty}^{\infty} dt\, \gamma(t)\, \eta(t) / [\lambda - i\eta(t)] \}. \tag{13}$$

Now, to find the characteristic functional for the process, x(t), subject to the initial condition, $x(0) = x_0$, use the solution of the stochastic differential equation, (letting γ be constant to make it easier to follow the derivation) in the definition,

$$C_x[\eta(\cdot)] = E\{ \exp[i\int_0^\infty \eta(t)\, x(t)\, dt] \}, \tag{14}$$

or,

$$= \exp[ix_0\int dt\ \eta(t)e^{-\gamma t}] \cdot \exp\{i\int dt\ \eta(t)e^{-\gamma t}\int dt' e^{\gamma t'}\ \theta(t-t')\ g(t') \},$$

where we have used the properties of the Heaviside unit step function, $\theta(t-t')$, so that all the integrations are over 0 -> ∞, thus making it easier to interchange the order of integration to give,

$$C_x[\eta(\cdot)] = \tag{15}$$

$$\exp[ix_0\int dt\ \eta(t)e^{-\gamma t}] \cdot \exp\{i\int dt'\ g(t')\ [\int dt e^{-\gamma(t-t')}\ \theta(t-t')\ \eta(t)] \}.$$

The second exponential term is now of exactly the same form as the characteristic functional for g(t), with the expression in square brackets replacing $\eta(t')$. Hence, making use of Eq. (13) we have,

$$C_x[\eta(\cdot)] = \exp[ix_0 \int_0^\infty dt\ \eta(t)e^{-\gamma t}] \ast \tag{16}$$

$$\exp \{i\int_0^\infty dt'\ \gamma\ [\int_{t'}^\infty dt\ e^{-\gamma(t-t')}\ \eta(t)]\ /\ [\lambda - i\ [\int_{t'}^\infty dt e^{-\gamma(t-t')}\eta(t)]\ ]\ \},$$

where we have used the properties of the step function.

To determine the marginal distribution of this process use $\eta(t) = \eta_0 \, \delta(t-t_0)$, which recovers the ordinary characteristic function for $x(t_0)$. A straightforward calculation yields,

$$C_{x(t_0)}[\eta_0] = \exp[i\eta_0 x_0 e^{-\gamma t_0}] \cdot \{ [\lambda - i\eta_0 \, e^{-\gamma t_0}] \, / \, [\lambda - i\eta_0] \}. \qquad (17)$$

This is the characteristic function of a r.v. that, with probability $\alpha = e^{-\gamma t}$, has the value $x_0 e^{-\gamma t}$, and with probability $(1-\alpha)$ is the sum of $x_0 e^{-\gamma t}$ and a r.v. with exponential distribution of parameter λ. This is the proof of Eq. (7). Setting $x_0 = 0$ and taking $t = h$ in Eq. (17), we have the characteristic function of the integrated shot noise, the second term in Eq. (6) corresponding to the innovation of the discrete EAR(1). As advertised, it is seen to have the value zero with probability α and to have an exponential distribution with probability $(1-\alpha)$. This completes the proof that our process leads back to EAR(1) when observed at equally spaced time intervals.

Now, taking $t_0 \to \infty$ Eq. (17) yields the marginal distribution of the steady state process,

$$C_{x(\infty)}[\eta_0] = \lambda / [\lambda - i\eta_0] \, , \qquad (18)$$

i.e. the characteristic function of an exponential distribution, as promised. The foregoing equations can all be generalized for when the decay parameter, γ, is not equal to the shot-noise rate, γ'. In that case the first appearance of γ in the second line of Eq. (16) is replaced by γ', yielding for the marginal distribution, conditional on $x(0) = x_0$,

$$C_{X(t_0)}[\eta_0] = \exp[i\eta_0 x_0 e^{-\gamma t_0}] \cdot \{ [\lambda - i\eta_0 e^{-\gamma t_0}] / [\lambda - i\eta_0] \}^{\gamma'/\gamma}. \quad (19)$$

One readily sees that as $t_0 \rightarrow \infty$ this becomes the characteristic function for a Gamma distribution, unless $\gamma' = \gamma$. In either case, it is apparent from the nature of the process that it is not time reversible, as is the output of an identical linear system, driven by Gaussian white noise (the O. U. process). In fact, it is just this shot noise driven process which is used as the first example of an irreversible, non-Gaussian process in reference [1].

Setting $t_0 = h$ in Eq. (19), it is clear that the second term,

$$\{ [\lambda - i\eta_0 e^{-\gamma h}] / [\lambda - i\eta_0] \}^{\gamma'/\gamma} ,$$

is the characteristic function for the EAR(1) innovation (the second term in Eq. 6) which, applied to Eq. (1), generates a Gamma marginal time series. Although the form of this characteristic function is difficult to interpret, it can be simulated in the following manner, based on the underlying continuous process. 1) Choose n from a Poisson distribution, mean $\gamma' h$ (n may be zero); 2) generate the n r.v.'s $\{t_i\}$ uniform on $[0, h]$; 3) generate n exponential (parameter λ) r.v.'s, E_i ; 4) let the innovation be

$$\xi = \sum_{i=1}^{n} e^{-\gamma t_i} \cdot E_i \; . \tag{20}$$

This agrees with a prescription given by Lawrance[9] for generating the innovation, also based on the underlying shot noise process.

V. Conclusion

We have shown the relationship between the irreversible and non-Gaussian continuous version of the EAR(1) model and the (reversible and Gaussian) Ornstein-Uhlenbeck process. There are a number of situations in which the continuous version of EAR(1) may be a reasonable model. The properties of non-negativity, irreversibility and a gamma marginal distribution may be appropriate for time series such as arise in geophysics. We have elsewhere proposed this process as a model for the hazard rate of components in a random environment.[10] In reference [10] the characteristic functional for the process is, in fact, applied directly to obtain the component and system reliability.

The Wiener and Ornstein-Uhlenbeck processes are often used as models of physical and economic phenomena because of their Gaussian properties, time-reversibility, and spectral densities. It will prove useful to enlarge the arsenal of continuous processes readily available for modeling phenomena which are non-Gaussian and irreversible but still possessing desirable spectral properties.

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Appendix A. The Multi-time Characteristic Function for the EAR(1)

To investigate certain properties of EAR(1) it is convenient to first derive the complete multi-time characteristic function for the process. This function is defined as,

$$C_x(\{\eta\}) = E\{\exp[\, i \sum_{n=-\infty}^{\infty} \eta_n x_n \,]\}. \tag{A1}$$

As is well known, we can write x_n as,

$$x_n = \sum_{k=0}^{\infty} \alpha^k \xi_{n-k} . \tag{A2}$$

Now, make the change of variable, $j = n - k$, and define the function $\theta_{jn} = 1$ if $j \geqslant n$, else 0. The latter is analogous to the Heaviside step function, and allows us to take both summations over $[-\infty, \infty]$, making it easier to interchange the orders of summation. The summation in Eq. (A1) can now be written,

$$\sum_{n=-\infty}^{\infty} \eta_n x_n = \sum_{j=-\infty}^{\infty} \xi_j \{ \sum_{n=-\infty}^{\infty} \eta_n \alpha^{j-n} \theta_{jn} \}. \tag{A3}$$

Denote the expression within the curly brackets by ω_j and insert in Eq. (A1). Hence, the characteristic function for the process, {x}, has been expressed in terms of the characteristic function for the innovation, {ξ},

$$C_x(\{\eta\}) = E\{\exp[\, i \sum_{j=-\infty}^{\infty} \omega_j \xi_j \,]\}. \qquad \text{(A4)}$$

Because the ξ_j are iid the expression on the right of this equation is simply the product of the characteristic functions for each ξ_j,

$$C_x(\{\eta\}) = \prod_j \{ (\lambda - i\omega_j \alpha)/(\lambda - i\omega_j) \}, \qquad \text{(A5)}$$

each term being the characteristic function for a r.v. having exponential distribution with probability (1-α), and the value zero with probability α. Replacing ω with its value from Eq. (A3), we have, finally,

$$C_x(\{\eta\}) = \prod_j \{ [\lambda - i(\sum_{n=-\infty}^{j} \eta_n \alpha^{j-n})\alpha] / [\lambda - i(\sum_{n'=-\infty}^{j} \eta_{n'} \alpha^{j-n'})] \}. \qquad \text{(A6)}$$

For example, if we are concerned with a finite observation period for this time series, where j = 0 -> N, we would simply set all η_j to zero except when $j \in [0, N]$.

All multi-time correlation functions can be recovered from this characteristic function in the usual manner by taking the appropriate derivatives wrt η_a, η_b, etc., and then setting all η's to zero. In addition, by suitable choices for the {η} we can obtain the characteristic function for certain averages of the {x} process. For example, setting η equal to η_0 for $j \in [0, 4]$ and zero elsewhere gives us the characteristic function for the average over 4 observations of the series. Alternatively, setting $\eta_j = \eta_{1j} + \eta_{2j}$, where η_{1j} is η_1 (a constant) for $j \in [0,4]$ and zero elsewhere, and setting η_{2j} equal to η_2 for $j \in [t, t+4]$ and zero elsewhere, yields the two time joint characteristic function for the moving averaged time series. As a final example, choose $\eta_n = \theta_{kn}(1-\beta)\beta^{k-n}$ (where $0 < \beta < 1$) yielding the characteristic function of the exponentially smoothed time series.

Bartlett[7] suggests a method for estimating the sampling statistics of the periodogram of a point process in continuous time using the characteristic functional for the process. We apply his method in the current context, letting,

$$\eta_j = \sqrt{2/N}\,\{\,\theta \exp[i\omega j] + \theta^* \exp[-i\omega j]\,\}, \tag{A7}$$

for $0 \leqslant j \leqslant N$, and zero elsewhere, where θ^* is the complex conjugate (c.c.) of θ. It is clear that we obtain the ordinary joint characteristic function of

$$J(\omega) = \sqrt{(2/N)}\sum e^{i\omega j}\, x_j,$$

and $J(-\omega)$, of the form,

$$C_{J,J^*}(\theta, \theta^*) = \prod_{j=0}^{N}\{\,\lambda - i\alpha[\cdot]\}\,/\,\{\,\lambda - i\,[\cdot]\}, \tag{A8}$$

where the summations can be carried out in closed form, yielding,

$$[\cdot] \quad = \quad [\, \theta\, (\alpha^{j} - e^{i\omega j}) \,/(\alpha - e^{i\omega}) \; + \; c.c. \,]. \tag{A9}$$

In any case, the various moments of the periodogram may be found from derivatives of the form,

$$(d/d\theta)^{p}\,(d/d\theta^{*})^{q}\;\; C_{J,J^{*}}(\theta,\, \theta^{*})\; \Big|_{\theta=0}\,.$$

Appendix B. The Second Order Linear System

It would be interesting to know whether a Gamma process also results if the same shot noise process, g(t), is filtered through a second order linear system. In this Appendix we answer that question in the negative by calculating the corresponding characteristic functional.

The stochastic differential equation of interest is that of a damped harmonic oscillator (or RLC circuit), driven by exponentially distributed shot impulses,

$$d^2/dt^2\, x \;+\; 2\gamma\; d/dt\; x \;+\; \omega_0^{\,2}\, x \;=\; g(t), \tag{B1}$$

where γ and ω_0 are the system damping constant and undamped, natural frequency, respectively. We have set the "mass" parameter equal to one. We will take advantage of the mechanical analogue to determine the response to the stochastic driving "force", g(t). As is well known there are three possible situations of physical interest, viz. over-damping, critical damping, and under-damping. With over-damping ($\gamma > \omega_0$) the effective circular frequency of the system is pure imaginary, hence there is no oscillatory component of system response and the response to a positive impulsive force damps out like $e^{-\gamma t}\sinh(\omega t)$. In this case the time series, x, is again non-negative, assuming g(t) is non-negative. With under-damping ($\gamma < \omega_0$) there is a damped, oscillatory response to a positive impulsive force. The response is like $e^{-\gamma t}\sin(\omega t)$. The process is then no longer non-negative. Finally, critical damping is the transition point between the other two cases, in which the response is like $(A + Bt)e^{-\gamma t}$, there is no oscillation, and the system passes through the zero point at most once. We will not consider this case here.

As is well known, the effective system frequency, ω, is determined by seeking the general solution to the homogeneous version of Eq. (B1) in the form, $A\, e^{i\delta_1 t} + B\, e^{i\delta_2 t}$. A standard calculation yields,

$$\delta_{1,2} = i[\, \gamma \pm \sqrt{\gamma^2 - \omega_0^2}\,]. \qquad \text{(B2)}$$

Define $\omega = \left|\sqrt{(\gamma^2 - \omega_0^2)}\right|$ for both under- and over-damped cases. Now, conservation of momentum for an impulsive force (of magnitude $z_j\delta(t)$) requires that the change of momentum ($\Delta p = m\dot{x}$) equals the total impulse, $\int F dt$. Integrating over the delta function yields,

$$\Delta p = z_j ,$$

or, since m = 1,

$$\dot{x}(0^+) = z_j.$$

Also, requiring that $x(0) = 0$ allows us to solve for A and B to match these initial conditions on $x(t)$, yielding,

$$x(t) = \frac{z_j}{\omega} e^{-\gamma t} \sin(\omega t),$$

for the under-damped case, and,

$$x(t) = \frac{z_j}{\omega} e^{-\gamma t} \sinh(\omega t),$$

for the over-damped case. Hence, using the linearity and translational invariance of Eq. (B1) we can write the general under-damped solution, assuming initial velocity, v_0, and position, x_0, as,

$$x(t) = x_0 \cos\omega t + \frac{v_0}{\omega} \sin\omega t + \sum_{j=1}^{n} \frac{z_j}{\omega} e^{-\gamma(t-t_j)} \sin(\omega t)\, \theta(t - t_j). \qquad \text{(B3)}$$

The cos and sin are replaced with cosh and sinh, respectively, for the over-damped case.

To calculate the characteristic functional for this process, we insert it into Eq. (12) in Section III and repeat (with slight modification) the steps following it, yielding (replace sin with sinh for overdamping),

$$C_x[\eta(\cdot)] = \exp\{i\int_0^\infty \eta(t)\,[x_0\cos\omega t + v_0/\omega\,\sin\omega t]e^{-\gamma t}dt\} \;*$$

$$\exp\{i\int_0^\infty dt'\,\gamma'\,[\int_{t'}^\infty \eta(t)\,\sin\omega(t-t')e^{-\gamma(t-t')}dt]/ [\omega\lambda - i\int_{t'}^\infty \eta(t)\sin\omega(t-t')e^{-\gamma(t-t')}dt]\} \qquad (B4)$$

Now, use $\eta(t) = \eta_0\delta(t - t_0)$ and let $t_0 \rightarrow \infty$ to obtain the ordinary characteristic function of the steady state marginal distribution of $x(t_0)$,

$$C_{x(t_0)}(\eta_0) = \exp\{\, i\int_0^\infty dt\,\gamma'\,\eta_0\,e^{-\gamma t}\sin(\omega t')/[\omega\lambda - i\eta_0 e^{-\gamma t}\sin(\omega t)]\}. \qquad (B5)$$

It is seen that even when γ' is a constant, and replacing the sin with sinh, this does not integrate to yield a characteristic function of a Gamma process. This can also be understood from another point of view. A straightforward analysis (we omit the details) shows that an overdamped second order system is equivalent to two first order systems in series. The output of the first of the first-order systems is, by our previous analysis, a Gamma process. However, feeding this into the second first-order system would not be expected to result in a Gamma process because only an exponentially distributed shot noise input can do that.