Reliability in a Gaussian Environment

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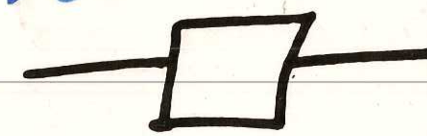
THE HAZARD RATE

- DEF'N. $h(t) = P_r\left[t < T \leq t+dt \mid T > t\right] / dt$



$$= \frac{f(t)}{1-F(t)}$$

$$\therefore \quad F(t) = 1 - e^{-\int_0^t h(t')\,dt'}$$

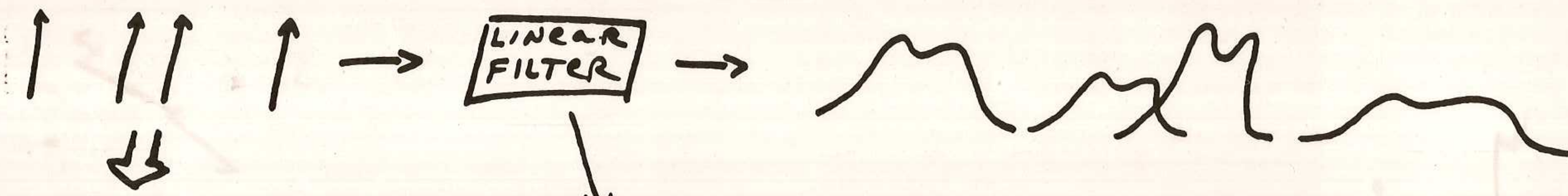
- RELIABILITY $= P_r[T > t]$

$$R(t) = 1 - F(t) = e^{-\int_0^t h(t')\,dt'}$$

$$\rightarrow \quad h(t) = -\frac{d}{dt}\ln R(t)$$

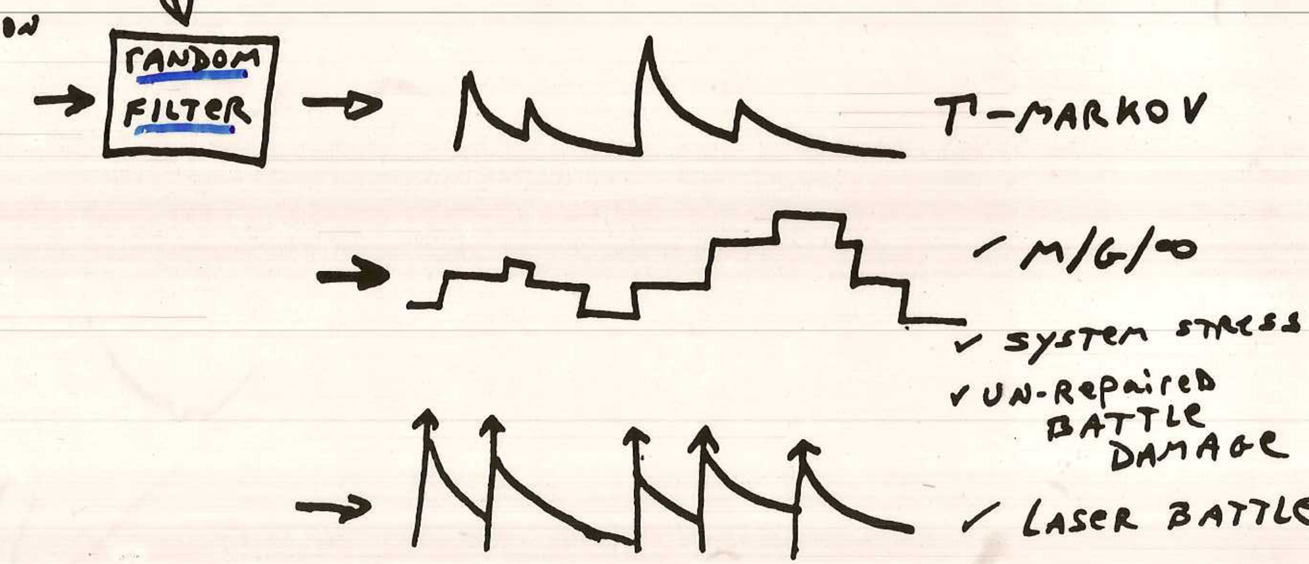
- Presumably averaged over
  - MFG VARIABLES
  - How it's operated
  - environmental cond's
  - ⋮

FILTERED SHOT NOISE PROCESSES

- 

SHOT NOISE/δ-FUNCTION SPIKES

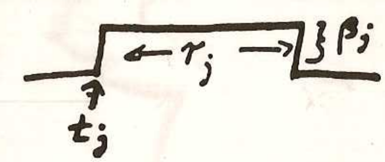
$$g(t) = \sum_{j=1}^{n} \delta(t - t_j)$$



- $h(t') = \lambda(t') + \sum_{j=1}^{n} r_j(t' - t_j)$

  ↑ random response function

  e.g. if failures ∝ unrepaired battle damage

  $r_j(t' - t_j) = \kappa \beta_j \{\Theta(t' - t_j) - \Theta(t - t_j - \tau_j)\} \sim$ 

- ALL ⇒ Non-exp'l lifetimes

GENERAL STOCHASTIC HAZARD PROCESS

- For j-th comp. $h_j(t) = \lambda_j(t) + g_j(t)$
  - $g_j(t)$: STOCH. PART
  - $\lambda_j(t)$: wearout/env. factors - indep. ∀ comp's.

- CONDITIONED on REALIZATION of $h(\cdot)$

$$R_h(t) = e^{-\int_0^t h(t')\,dt'}$$

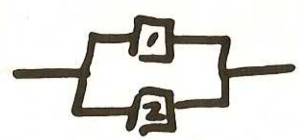
- FOR 1-COMP. → SIMPLY AVE. OVER SAMPLE PATHS

$$R(t) = E_h\,[R_h(t)] = E\left\{ e^{-\int_0^t h(t')\,dt'} \right\}$$

  - Define effective hazard rate

$$\hat{h}(t) = -\frac{d}{dt}\ln R(t) \qquad \left(\neq E\{h(t)\} \right)$$

- e.g. 2-comp's in // :

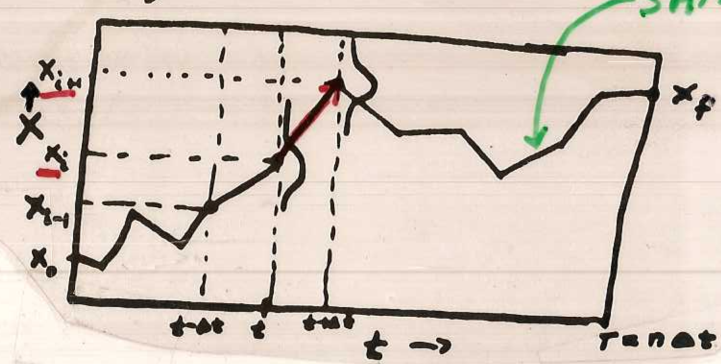


$$R_h(t) = 1 - (1 - R_{h_1}(t))(1 - R_{h_2}(t)) = R_{h_1}(t) + R_{h_2}(t) - R_{h_1} \cdot R_{h_2}$$

$$\therefore\ R(t) = R_1 + R_2 - E\left\{ e^{-\int_0^t (h_1 + h_2)\,dt'} \right\} \neq R_1 + R_2 - R_1 \cdot R_2$$

SHORT-TIME LIMIT OF SOLUTION: SDE

- $\hat{P}_{rd}[x_i+\Delta x, t+\Delta t \,/\, x_i, t] = \dfrac{[1+\hat{\alpha}\, g'(x_i)\Delta t]}{\sqrt{2\pi\sigma^2\Delta t}} \exp\left\{-\dfrac{(\Delta x + g(x_i)\Delta t)^2}{2\sigma^2\Delta t}\right\}$
  - pdf of X_{i+1}
  - Dropped terms in Δt of higher order
  - $0 \le \hat{\alpha} < 1$



- Probability of a 'path'
  - really a prob. density functn.
  - in limit as $\Delta t \to 0 \Rightarrow$ prob. density functional of a continuous path
  - $= \prod_i \hat{P}_{rd}[\underbrace{x_i+\Delta x}_{X_{i+1}}, t_i+\Delta t \,|\, x_i, t_i]$ ✓ USED MARKOV property ✓ CONDITIONAL ON x_0

A Transformation on Path Integrals

$\Delta y + \text{`}f(y)\text{'}\Delta t = \sqrt{g(y)}\,\Delta w$ Stochastic Differential Equation (SDE)

↑ environment ↑ Wiener process

-- Multiplicative noise term of form $\sqrt{g(x)}\Delta w$ can be transformed with appropriate stochastic calculus

-- stochastic differential equation: $\Delta x + \text{`}g(x)\text{'}\,\Delta t = \Delta w$

"-force"

-- ∃ ambiguity wrt 'g(x)' even though considering additive noise term

-- $\Delta x \equiv x_{i+1} - x_i$; $\Delta w \equiv w_{i+1} - w_i$; w = Wiener process

-- Possible "discretizations": $\text{`}g(x)\text{'} = (1 - \hat{\alpha})\, g(x_i) + \hat{\alpha}\, g(x_{i+1})$

-- $\hat{\alpha} = 1/2$ => Stratonovich SDE; $\hat{\alpha} = 0$ => Ito SDE

-- since $\sigma(x)$ = const. (possibly after a transformation)

Fokker-Planck eq'n is same [but may be relevant for path integral]

Short-Time Propagator for Paths

-- require short time conditional pdf for
Chapman-Kolmogorov eq./path integral limit --> GAUSSIAN for $\Delta t \to 0$

-- Expand SDE in Taylor series about x_i :

$$\Delta x + [\, g(x_i) + \hat{\alpha}\, g'(x_i)\Delta x + o(\Delta x)\,]\Delta t = \Delta w$$

$$\Delta x\,[\, 1 + \hat{\alpha} g'(x_i)\Delta t\,] + g(x_i)\Delta t = \Delta w$$

$$\Rightarrow \quad \Delta x = -\, g(x_i)\Delta t/[\, 1 + \hat{\alpha} g'(x_i)\Delta t\,] + \Delta w/[\, 1 + \hat{\alpha} g'(x_i)\Delta t\,]$$

-- Hence, conditional on x_i, for short times, x_{i+1} will have
mean $E(x_{i+1}) \approx x_i + g(x_i)\Delta t/[\, 1 + \hat{\alpha} g'(x_i)\Delta t\,]$

-- and Var$(x_{i+1}) \approx \sigma^2 \Delta t\, /\, [\, 1 + \hat{\alpha} g'(x_i)\Delta t\,]^2$

THE TRANSFORMATION FORMULA

- Express Expectations of FUNCTIONALS - over SAMPLE paths of SDE
  - IN TERMS OF Expectations over SAMPLE PATHS OF WIENER (Brownian MOTION) process

- $$E_{x_0,x_f}\{F[x(\cdot)]\}_{\underline{SDE}} = E_{x_0,x_f}\left\{F[x(\cdot)]\, e^{\hat{\alpha}\int_0^t g'(x)dt - \int_{x_0}^{x_f}\frac{g(x)dx}{\sigma^2} - \int_0^t \frac{g^2(x)dt}{2\sigma^2}}\right\}_{\underline{W}}$$

  Stratonovich (ordinary) Calculus

  - choose $\hat{\alpha} = \frac{1}{2}$ for convenience $\Rightarrow$ Stratonovich (ordinary) Calculus (NOT really necessary)
  - Let $G'(x) \equiv g(x)$

- $$\boxed{E_{x_0,x_f}\{F[x(\cdot)]\}_{SDE} = e^{\frac{G(x_0)-G(x_f)}{\sigma^2}}\, E\left\{F[x(\cdot)]\, e^{\frac{1}{2}\int_0^t dt\left[g'(x) - \frac{g^2}{\sigma^2}\right]}\right\}_{W}}$$

APPLICATIONS OF THE TRANSFORMATION

- Let $F[x(\cdot)] = 1$, $g(x) = \Lambda x$ (ORNSTEIN-UHLENBECK PROCESS)

$$LHS = \hat{P}_{rd}[x_f, t \mid x_0, 0] = \frac{\Lambda}{\pi\sigma^2(1-e^{-2\Lambda t})} \exp\left\{ \frac{-\Lambda(x_f - x_0 e^{-\Lambda t})^2}{\sigma^2(1-e^{-2\Lambda t})} \right\}$$

& $$RHS = e^{\frac{x_0^2 - x_f^2}{2\sigma^2} + \frac{\Lambda}{2}t} * \underset{x_f}{E}\left\{ e^{-\frac{\Lambda^2}{2\sigma^2}\int_0^t x^2(t)\,dt} \right\}_w$$

⟹ a little algebra + translate into cosh & sinh

Gives a now classic formula of Cameron/Martin (KAC FUNCTIONAL)

- Now, identify hazard rate function as $\propto X^2(t)$, $X \sim$ Wiener Process, w

$$\therefore R(t) = \int dx_f \underset{x_f}{E}\left\{ e^{-\alpha\int x^2(t)\,dt} \right\}_w = E\left\{ e^{-\int_0^t h(t)\,dt} \right\}$$

APPLICATIONS TO ORNSTEIN-UHLENBECK PROCESS

- $h(t) = \alpha x^2(t)$, $x \sim$ O.U. Process

$$E_{x_0, x_f}\left\{ e^{-\alpha\int_0^t x^2(t)\,dt} \right\}_{O.U.} = e^{\frac{x_0^2 - x_f^2}{\sigma^2} + \frac{\Lambda}{2}t} \; E_{x_0, x_f}\left\{ e^{-\left[\frac{\Lambda^2}{2\sigma^2} + \alpha\right]\int_0^t x^2(t)\,dt} \right\}_{W}$$

  - $E\{\cdot\}$ on RHS is NOW SAME as above
  - set $x_0 = 0$ & $\int dx_f$ to obtain $R(t)$

- OFFSET O.U. : Constant Force/VOLTAGE $\quad L\frac{di}{dt} + Ri = \xi(t) + V_0$

  i.e. $\Delta x + \Lambda(x - \hat{x})\Delta t = \Delta w$

  ✓ use $\hat{x} = \frac{1}{2}\sqrt{Var(O.U.)}$ to see effect of $\hat{x}$ in SIMULATION

AVERAGES OVER PATHS

- CALCULATE Expected values of <u>function</u> $F[\{x_i\}] = F[x_0, x_1, x_2, \cdots, x_n]$

$$E\left\{F(\{x_i\}_{i=0}^{n})\right\} = \int dx_1 \cdots \int dx_{n-1} \prod_i \hat{P}_{rd}[x_{i+1}, t_{i+1} \mid x_i, t_i]\, F(\{x_i\}_{i=0}^{n})$$

- TAKE The Limit $\rightarrow$ continuous paths : $x_0 \rightarrow x_f$
  - insert our short-time pdf
  - $F[\,] \rightarrow$ functional of paths

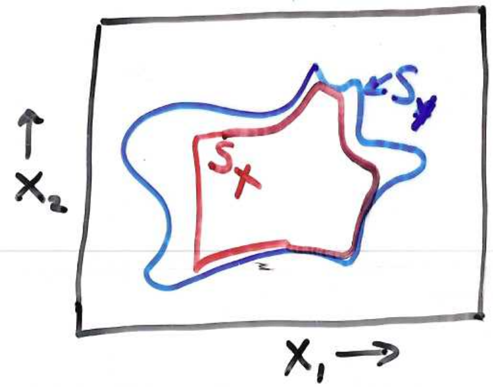
$$\xrightarrow[n\to\infty]{\Delta t \to 0} \int \mathcal{D}x(\cdot)\, F[x(\cdot)] \exp\left\{ \hat{\alpha}\int_0^t g'(x)\,dt - \int_0^t \frac{\dot{x}^2}{2\sigma^2}\,dt - \int_0^t \frac{g(x)\,dx}{\sigma^2} - \int_0^t \frac{g^2\,dt}{2\sigma^2} \right\}$$

$\mathcal{D}x(\cdot) = \lim_{\Delta t \to 0} \prod_i^{n-1} \frac{dx_i}{\sqrt{2\pi\sigma^2 \Delta t}}$ $\longleftarrow$ usual Wiener measure

ANALOGUE OF OUR PROCEDURE

- $\vec{Y} \sim$ pdf $g_{\bar{Y}}(\bar{y})$: Support S_y
  $\vec{X} \sim$ pdf $f_{\bar{X}}(\bar{x})$: Support S_x

$S_x \subseteq S_y$



- $$E_{\underline{x}}\{\hat{F}(\vec{X})\} = \int_{S_x} \hat{F}(\bar{x})\, f(\bar{x})\, d^k x$$

$$= \int_{S_x} \hat{F}(\bar{x})\, \frac{f_x(\bar{x})}{g_y(\bar{x})} \cdot g_y(\bar{x})\, d^k x$$

$$= E_{\underline{y}}\left\{\hat{F}(\vec{Y})\, \frac{f_x(\bar{Y})}{g_y(\bar{Y})} \cdot 1_{[\bar{Y} \in S_x]}\right\} \qquad (1)$$

- **If** $S_x = S_y \Rightarrow$

$$= E_{\underline{y}}\left\{\hat{F}(\vec{Y})\, \frac{f_x(\bar{Y})}{g_y(\bar{Y})}\right\} \qquad (2)$$

➡ Don't need to calculate JACOBIAN if $g_y(\bar{y})$ known, Eq (2) helpful.

- If $\nexists$ $\bar{x} \leftrightarrow \bar{y}$ (1-1), using Eq. (1) may be helpful.
  (because $S_x \neq S_y$)

THE (WIENER PROCESS)^2

- $h(t) = \lambda(t) + \alpha\, X^2(t)$ where $X(t) \sim$ Wiener Process

- Require:

$$C_{X^2}[-i\,\eta(\cdot)] = E\left\{ e^{\alpha \int^t \eta(t')\, X^2(t')\, dt'} \right\}$$

- CAMERON & MARTIN (BULL. AM. MATH. SOC. 1945; TRANS " " " 1949)

— EVALUATED USING LINEAR TRANSFORMATIONS OF SPACE OF SAMPLE PATHS, $X(\cdot)$, ONTO ITSELF

-- related to sol'n of certain STURM-LIOUVILLE D.E.'s

$$f''(t) + \alpha\, \eta(t)\, f(t) = 0 \qquad \ni \qquad f_\lambda(0) = f'(t) = 0$$

$$\Rightarrow \quad C_{X^2}[-i\,\eta(\cdot)] = \left[\frac{f_\alpha(t)}{f_\alpha(0)}\right]^{1/2}$$

where ✓ $-\infty < \alpha < \alpha_0$

✓ $f_\alpha(\cdot)$ depends on $\eta(\cdot)$

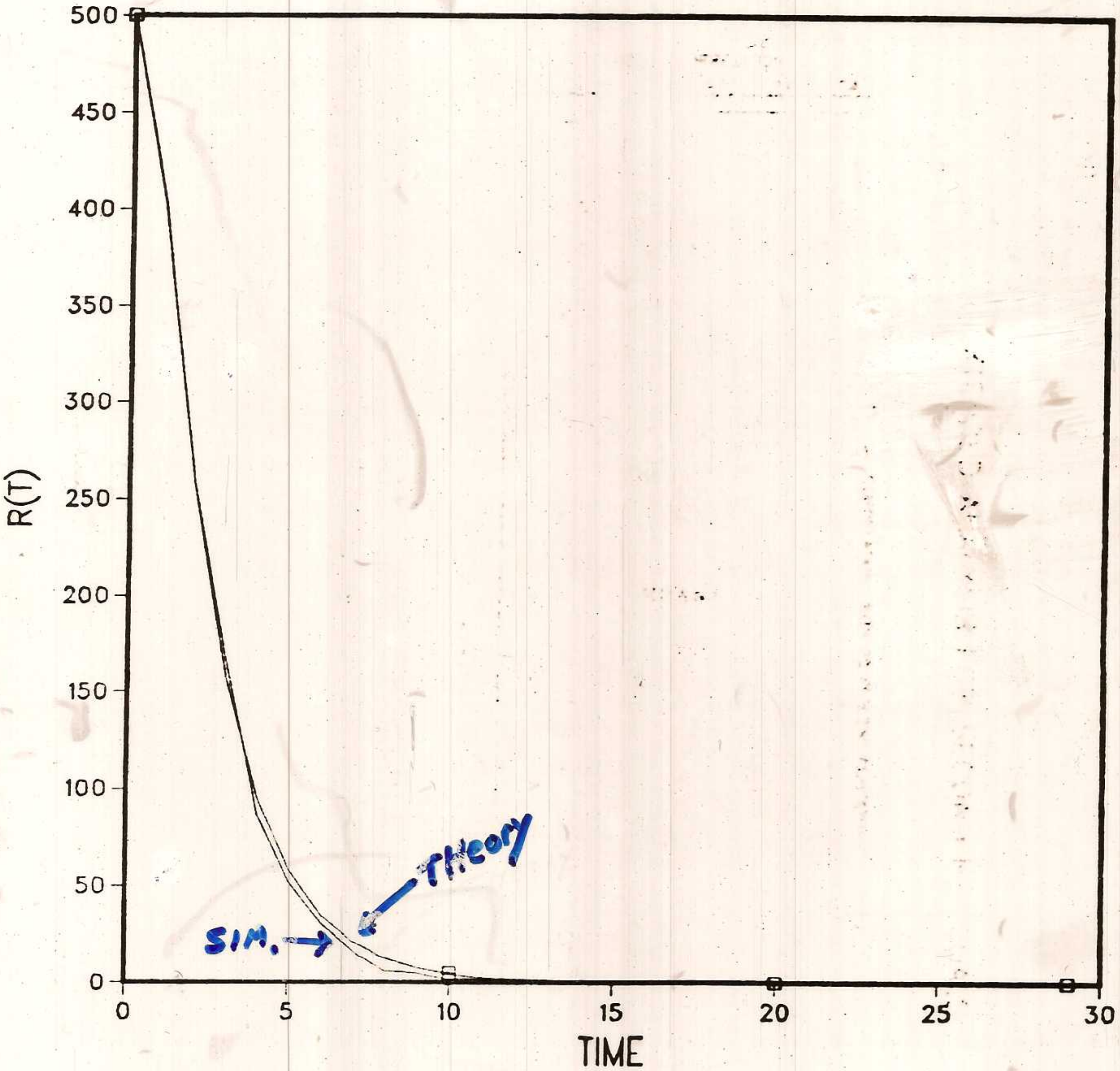
- For $\eta = -1$ C & M showed

$$C_{X^2}[i] = \left[\frac{1}{\cosh(\sqrt{\alpha}\, t)}\right]^{1/2}$$

- $$\left[R(t) = e^{-m\lambda t} \frac{1}{\sqrt{\cosh(\sqrt{m\alpha}\, t)}} \right]$$ For m-comp's in series

RELIABILITY

(WIENER PROCESS)^2



$h(t) = \alpha x^2$

$x \leftarrow x + \sqrt{2 \cdot D \cdot dt} * N$

IF $(U \leq \alpha x^2 dt) \Rightarrow$ FAIL

$dt = .01$

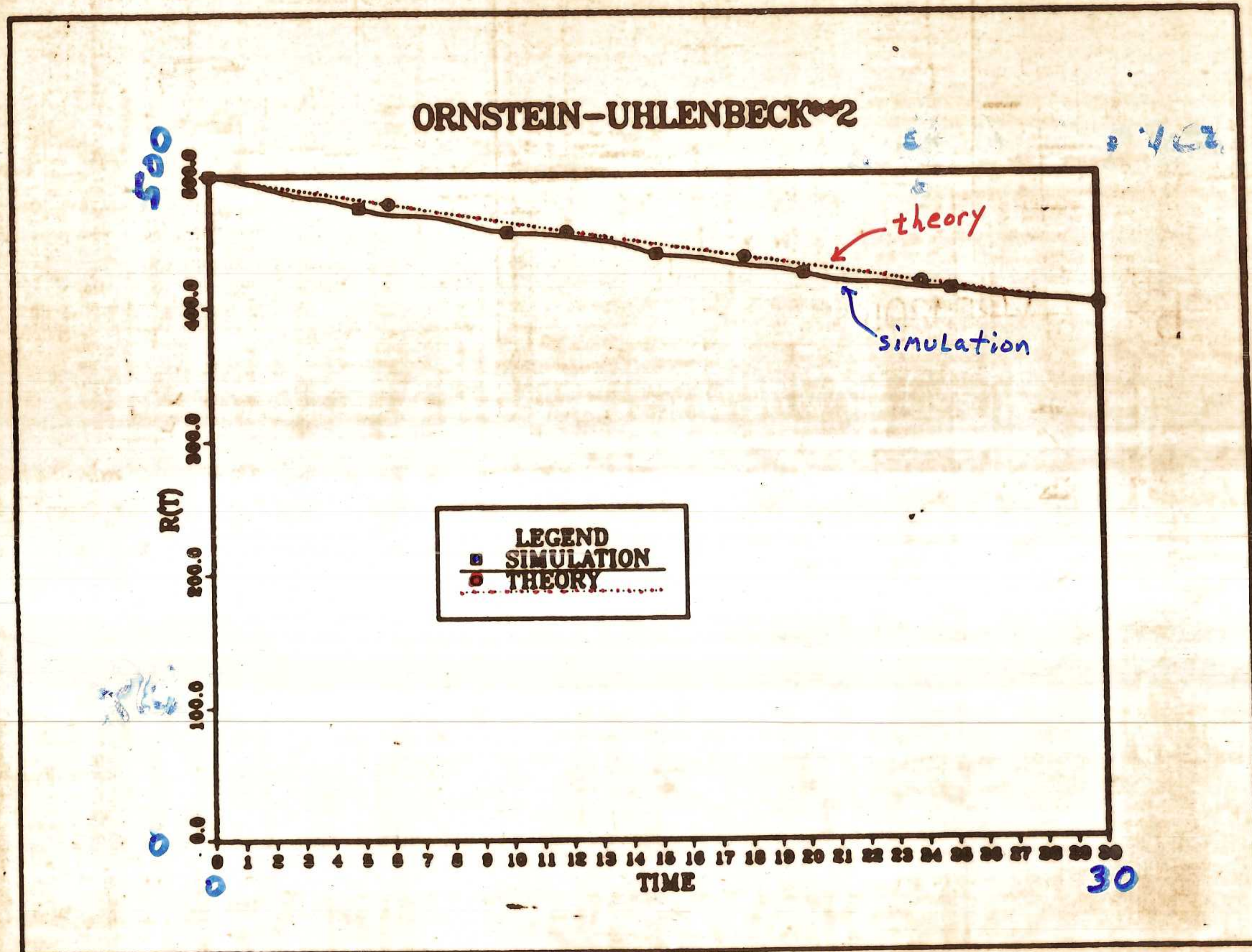
$\sigma^2 = 2D = 1/2$

$\alpha = 1.$

STATIONARY-GAUSSIAN ENVIRONMENT

- ORNSTEIN-UHLENBECK PROCESS
  - $\dot{x} + \Lambda x = \eta(t)$ e.g. $L\frac{di}{dt} + Ri = V(t)$
  - $P(x,t/x_0,t_0) = \left[\frac{\Lambda}{\pi\sigma^2(1-e^{-2\Lambda t})}\right]^{1/2} \exp\left\{-\frac{\Lambda}{\sigma^2}\frac{(x-x_0e^{-\Lambda t})^2}{(1-e^{-2\Lambda t})}\right\}$
- Hazard rate, $h(t) = \alpha\, x^2(t)$
- $R(t) = E^{av}\left\{e^{-\alpha\int_0^t x^2\,dt}\right\}_{x_0=0}$

$$R(t) = \frac{e^{\frac{\Lambda}{2}t}\,(\Lambda^2+2\sigma^2\alpha)^{1/4}}{\left[\Lambda\sinh(\sqrt{\Lambda^2+2\sigma^2\alpha}\,t) + \sqrt{\Lambda^2+2\sigma^2\alpha}\,\cosh(\sqrt{\Lambda^2+2\sigma^2\alpha}\,t)\right]^{1/2}}$$



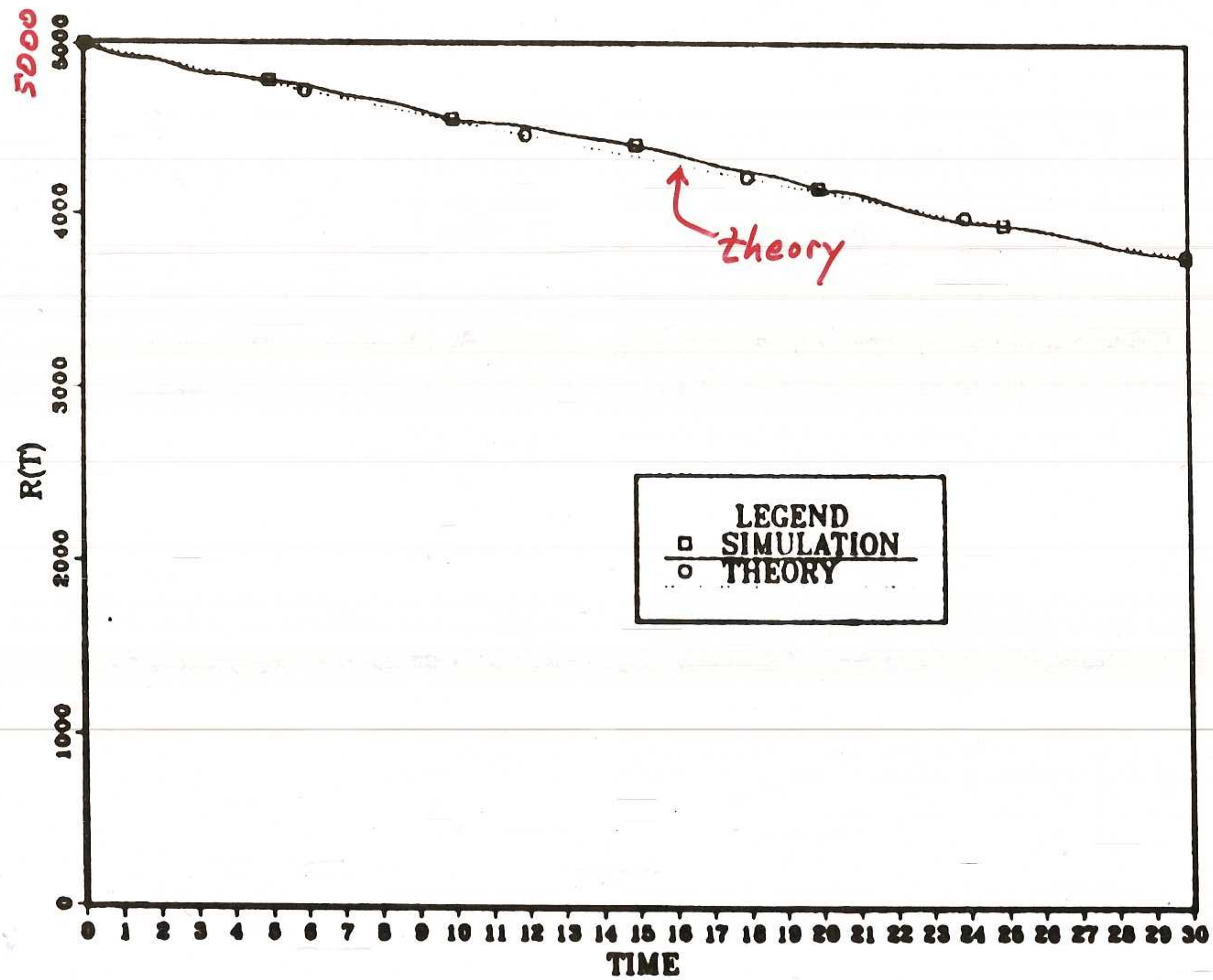
$$X \leftarrow X e^{-\Lambda \cdot dt} + \sqrt{2 \cdot D \cdot dt}\, N$$

$dt = .003$

$\sigma^2 = \frac{1}{2} = 2D$

$\Lambda = 33.3$

OFFSET ORNSTEIN-UHLENBECK**2



$$\Delta X + \Lambda (X - \hat{X})\Delta t = \Delta w \qquad \hat{X} = \frac{1}{2}\sqrt{Var\,(O.U.)}$$