Reliability in a Random Environment

Edward B. Rockower
Naval Postgraduate School
Dept. of Operations Research
Monterey, CA. 93943

ABSTRACT

Correlations in the failures of subsystems or components may arise when they share a common, random environment. We show that a natural tool for calculating the reliability of such systems is the characteristic functional of the random hazard rate, h(t). Some general results for the reliability of series and parallel systems in terms of the characteristic functional of the hazard rate are derived and applied to a number of models of random environments. The applications include random hazard functions arising from 1) non-fatal shocks of random amplitudes, 2) a Markovian, Gamma-marginal stochastic process, 3) system stress related to un-repaired damage from incoming rounds, and 4) impulsive and accumulated heat stresses from a laser battle.

1. INTRODUCTION

Consider a system whose components are subject to a random environment[1], possibly including random shocks. In general, the i^{th} component hazard rate, $h_i(t)$, may be written as the sum of two parts. The first (deterministic) part, $\lambda_i(t)$, accounts for wearout and those random failures which occur independently for each component. The second (stochastic) part, $g_i(t)$, arises from those environmental conditions which are shared by two or more components in the system.

We develop a general formalism for calculating the reliability of a system in a random environment in terms of the characteristic functional of the hazard rate. Various models of random environments are then proposed and the corresponding expressions for the characteristic functionals are calculated. The reliability of systems in such environments are then found using these characteristic functionals. The equivalent, deterministic hazard functions for single components in some of these models are found not to be constant in time[2].

In Section II the general formalism is introduced and illustrated for a single component subject to an environment which includes non-fatal shocks with random amplitudes. In Sections III and IV the reliability of series and parallel systems is presented in general terms and applied to the non-fatal shock model. Section V presents a model for the hazard rate as a Markovian, Gamma-marginal stochastic process. Finally, in Section VI we present three models in which the failure rate is proportional to system stress. Each of these models can be applied to multi-component systems using the methods developed in the earlier Sections.

II. One Component

The simplest case is that of one component with hazard rate $h(t)$, a non-negative stochastic process. The reliability of the component, subject to a particular realization of the environment (hence, $h(t)$) is defined as,

$$R_h(t) = \Pr[T > t]_h, \tag{1}$$

where T is the random variable (r.v.) equal to the time the component fails and the subscript denotes a particular realization of the stochastic process, h. Using the usual definition of the hazard rate[3], this is,

$$R_h = \exp\left[-\int_0^t h(t')dt' \right]. \tag{2}$$

Hence, averaging over the random environments we have,

$$R(t) = E_h[R_h] = E\left\{ \exp\left[-\int_0^t h(t')dt' \right] \right\} \tag{3}$$

or,

$$R(t) = C_h[\eta(\cdot)] \Big|_{\eta(\cdot)=i} , \tag{4}$$

where,

$$C_{h,t}[\eta(\cdot)] = E\left\{ \exp\left[i\int_0^t \eta(t')\, h(t')dt' \right] \right\}, \tag{5}$$

defines the characteristic functional[4,5] of the hazard rate. In other words, this characteristic functional, evaluated for a particular value of the test function, $\eta(t)$, yields directly the reliability of a component. We will see below how this generalizes to multi-component systems in a random environment.

We can write the reliability of a component or system in terms of an effective, deterministic hazard rate, $\hat{h}(t)$, using the usual definition of hazard rate,

$$\hat{h}(t) = -\, d/dt \;\ln\{ R(t) \},$$

but we must remember not to use this "hazard rate" to calculate reliability of larger aggregations, unless such aggregations are composed of components or subsystems not sharing a common, random environment.

Deterministic hazard rate plus random shocks

If the environment gives rise to non-fatal random shocks with rate $\gamma(t)$, then the hazard rate may be represented by a deterministic function, $\lambda(t)$, plus delta function spikes, possibly with random amplitudes. For convenience we will sometimes replace $\lambda(t)$ or $\gamma(t)$ with constants. However, in all the reliability formulas derived here, λt can always be replaced with $\int\lambda(t')dt'$ for non-constant background hazard rate, and γt can be replaced with $\int\gamma(t')dt'$ for a non-homogeneous shock process. Note that $\lambda(t)$ may be really an effective hazard rate, obtained from the preceding equation, when one portion of the random hazard rate is unique to each individual component, i.e. not shared among separate components.

The hazard rate for non-fatal shocks occurring at times t_j is,

$$h(t') = \lambda + \sum_{j=1}^{n} \alpha_j \,\delta(t' - t_j), \tag{6}$$

where the pdf of the t_j is $\gamma(t)/\bar{n}$, $\bar{n} = \int_0^t \gamma(t')dt'$, and n is a Poisson r.v. with mean $\bar{n}$.

The characteristic functional of this process, derived in Appendix A, is,

$$C_{h,t}[\eta(\cdot)] = \exp\{ i\lambda \int_0^t \eta(t')dt' + \int_0^t \gamma(t') [C_\alpha(\eta(t')) - 1]dt'\}, \qquad (7)$$

where C_α is the ordinary characteristic function of the shock amplitudes, α_j. In particular, if the shock amplitudes have an exponential distribution (parameter δ) then,

$$C_\alpha(\omega) = \delta/[\delta - i\,\omega], \qquad (8)$$

and,

$$C_{h,t}[\eta(\cdot)] = \exp\{ i\lambda \int_0^t \eta(t')dt' + i\int_0^t \gamma(t')\, \eta(t')/[\delta - i\eta(t')]\, dt'\}. \qquad (9)$$

Hence, setting $\eta(t') = i$, the reliability is,

$$R(t) = \exp\{ -\lambda t - \bar{n}\,/[\delta + 1] \}, \qquad (10)$$

or, assuming a stationary environment (γ = const.) and using $\bar{\alpha} = 1/\delta$,

$$R(t) = \exp\{ -\lambda t - \gamma t\, \bar{\alpha}\,/[\bar{\alpha} + 1] \}. \qquad (11)$$

The effective hazard rate, $\hat{h}(t)$, is constant for this model. Note that as $\bar{\alpha} \to \infty$ the rate of failure becomes λ plus the rate, γ, of occurrence of (fatal) shocks (in fact, λ could be a background rate of infinitely high shocks).

III. Two different components in the same environment

If two or more components are in different, independent environments the system reliability follows from the usual formulas[3] expressed in terms of the individual reliabilities. We consider here two different components seeing the same environment. The reliability of two components in series subject to a given realization of the environmental conditions is,

$$R_h(t) = P_h[T > t] = R_{h_1}(t) \cdot R_{h_2}(t). \tag{12}$$

If the components are in a parallel, redundant system the reliability is,

$$R_h(t) = 1 - [1 - R_{h_1}(t)] \cdot [1 - R_{h_2}(t)] \tag{13}$$

$$= \exp[-\int h_1(t')dt'] + \exp[-\int h_2(t')dt'] - \exp\{ -\int [h_1(t') + h_2(t')] dt' \},$$

where all integrals are 0 -> t. Hence, averaging over realizations of the environment we have, using the definition of the characteristic functional,

$$R(t) = \tag{14}$$

$$C_{h,t}[\eta(\cdot)] \,|_{\eta(\cdot)=i} + C_{h,t}[\eta(\cdot)] \,|_{\eta(\cdot)=i} - C_{h_1+h_2,t}[\eta(\cdot)] \,|_{\eta(\cdot)=i}$$

Clearly, the last term also represents the reliability of a series circuit with the same two components. In the following it will be understood that all characteristic functionals are for the processes over the interval [0, t].

As an example, consider two components in series, seeing the same shocks, but experiencing different amplitudes, α, $k\beta$; where α and β are i.i.d. exponential r.v's. Then $h_1 + h_2$ is given by

$$h(t') = \lambda_1 + \lambda_2 + \sum_{j=1}^{n} (\alpha_j + k\beta_j)\, \delta(t' - t_j). \tag{15}$$

In this case C_α in Eq. (7) is replaced by the characteristic function of the sum of two exponentials,

$$C_\alpha(\omega) = \delta/[\delta - i\omega] \cdot \delta/[\delta - i\omega k], \tag{16}$$

and λ is replaced by $\lambda_1 + \lambda_2$, yielding,

$$C_{h_1+h_2}[\eta(\cdot)] = \exp\{ i(\lambda_1+\lambda_2)\int_0^t \eta(t')dt' \} \cdot \exp\{ \int_0^t \gamma(t')dt' [\, \delta/[\delta - i\eta(t')] \cdot \delta/[\delta - i\eta(t')k] - 1 \,] \}. \tag{17}$$

The reliability of the series system is then found by substituting $\eta(t') = i$, yielding,

$$R_{series}(t) = \exp\{ -(\lambda_1+\lambda_2)t - \gamma t[\, 1 - \delta^2/[(\delta + 1)(\delta + k)] \,] \}. \tag{18}$$

The parallel system reliability is given by,

$$R_{||}(t) = R_1(t) + R_2(t) - R_{series}(t), \qquad (19)$$

where $R_{1,2}$ are the same as Eq. (10) with δ replaced with δ and δ/k, respectively.

If $\alpha = \beta$ in the above, i.e. the amplitude of shocks seen by the two components are proportional, then we have,

$$h_1(t') + h_2(t') = \lambda_1 + \lambda_2 + (1+k)\sum_{j=1}^{n} \alpha_j\, \delta(t' - t_j). \qquad (20)$$

This is the same as for a single component, except $\lambda_0 \to \lambda_1 + \lambda_2$ and $\delta \to \delta/(1+k)$, hence,

$$C_{h_1+h_2}[\eta(\cdot)] = \exp\{\, i(\lambda_1+\lambda_2)\int_0^t \eta(t')dt' \,\} \cdot \qquad (21)$$

$$\exp\{\, \int_0^t \gamma(t')dt' [\, \delta/[\delta - i\eta(t')\,(k+1)] - 1 \,] \,\},$$

and the reliability of a series circuit is,

$$R_{series}(t) = \exp\{ -(\lambda_1 + \lambda_2)t - \gamma t\, [(k+1)/(\delta+k+1)] \}. \qquad (22)$$

The reliability of a parallel circuit is again given by Eq. (19).

IV. m-Identical components

For m components in series, subject to a given realization of the environment, the reliability is,

$$R_h(t) = R_{h_1}(t)\, R_{h_2}(t)\, R_{h_3}(t) \cdots$$

$$= \exp\{ -\int_0^t (h_1 + \cdots + h_m)dt' \}. \tag{23}$$

Hence, averaging over the environment, the reliability is,

$$R(t) = C_{h_1+h_2+\ldots}[\eta(\cdot)]\Big|_{\eta(\cdot)=i}\,. \tag{24}$$

a) If all components respond to a given shock with independent amplitudes we obtain,

$$R_a(t) = \exp\{ -m\lambda_0 t + \gamma t\, [\delta^m/(\delta+1)^m - 1] \}, \tag{25}$$

b) if the hazard rate is exactly the same for all components,

$$R_b(t) = \exp\{ -m\lambda_0 t + \gamma t\, [\delta/(\delta+m) - 1] \}. \tag{26}$$

The effective hazard rate, $\hat{h}(t)$, is again constant. Note that

$$(1+1/\delta)^m = 1 + m/\delta + \cdots,$$

hence,

$$(1+1/\delta)^m > 1 + m/\delta,$$

or,

$$\delta^m/(\delta+1)^m < \delta/(\delta+m),$$

which implies that $R_a < R_b$. In other words, as expected, the reliability of a series system is higher in the more highly correlated environment. Also, using Eq. (19), it is clear that the reliability of a parallel system will be lower in a more highly correlated environment.

m-Identical components in parallel

The reliability of m-identical components in a parallel redundant system subject to a given realization of the environment is,

$$R_h(t) = 1 - \prod_{j=1}^{m} [\, 1 - \exp\{-\int_{o}^{t} h_j(t')dt'\}\,]. \tag{27}$$

This can be expanded using binomial coefficients, $C^m{}_j$, as,

$$R_h(t) = 1 - \sum_{j=0}^{m} (-)^j\, C^m{}_j\; 1 \cdot R_{h_1}(t) \cdots R_{h_j}(t). \tag{28}$$

Hence, averaging over the environment, we have,

$$R(t) = 1 - \sum_{j=0}^{m} (-)^j\, C^m{}_j\; e^{-j\lambda_0 t} \exp\{\, \gamma t\, [\delta^j/(\delta+1)^j - 1]\, \}, \tag{29}$$

for independent response to the shocks, and,

$$R(t) = 1 - \sum_{j=0}^{m} (-)^j\, C^m{}_j\; e^{-j\lambda_0 t} \exp\{\, \gamma t\, [\delta/(\delta+j) - 1]\, \}, \tag{30}$$

if there is exactly the same hazard rate for all components (each component sees exactly the same amplitude shocks).

In general, for m identical components in parallel, all with the same hazard rate, $h(t) = \lambda_0(t) + g(t)$ (where $g(t)$ is random),

$$R(t) = 1 - \sum_{j=0}^{m} (-)^{j} C^{m}_{j} \cdot \exp\{ -j\int \lambda_0(t')dt'\} \cdot C_g[\eta(\cdot)]\big|_{\eta(\cdot)=j\cdot i} \qquad (31)$$

From the foregoing it is clear how to generalize the formulas for reliability in a random environment to more general configurations containing components both in series and in parallel:

1) using the usual rules for probabilities, write the reliability of the system in terms of the individual component hazard processes, conditioned on a given realization of the environment, hence of the $\{h_j(t)\}$ (cf. Eq. 27),

2) average over the environment, hence over the $\{h_j(t)\}$,

3) express the result in terms of the appropriate characteristic functionals (cf. Eq. 31), and finally,

4) obtain the effective component or system hazard rate, $\hat{h}(t)$.

V. Exponential/Gamma Hazard rate

Consider an environment giving rise to a Markov hazard rate process with a Gamma marginal[1,6,7] distribution which is common to all components in the system of interest. We show in Appendix B that the characteristic functional for such a process is,

$$C_g[\eta(\cdot)] = \exp\{\upsilon\int_0^t dt'\, \gamma(t')[\delta/[\delta - i\int_{t'}^{t} dt''\eta(t'')\exp(-\int_{t'}^{t''}\gamma(\tau)d\tau)] - 1\,]\}. \qquad (32)$$

The correlation function for this process in the stationary case (γ const.) is $\rho = e^{-\gamma\tau}$. When $\upsilon=1$ the process is exponential.

The reliability of a single component with hazard rate $h(t) = \lambda(t) + g(t)$ is then,

$$R(t) = \exp\{-\int\lambda(t')dt'\} \cdot C_g[\eta(\cdot)]\Big|_{\eta(\cdot)=i} . \qquad (33)$$

carrying out the resulting integrations yields,

$$R(t) = \qquad (34)$$

$$\exp\{-\int\lambda(t')dt' - \upsilon\gamma t/(\gamma\delta+1)\} \cdot \{\gamma\delta/(\gamma\delta+1 - e^{-\gamma t})\}^{\upsilon(\gamma\delta+2)/(\gamma\delta+1)} .$$

The effective hazard rate, $\hat{h}(t)$, is clearly not constant for this model. This can be generalized to multi-component systems with the methods from the previous sections. For example, the reliability of m identical components in series is obtained by replacing $\lambda \to m\lambda$ and using $\eta(\cdot) = m \cdot i$ in the characteristic functional. From Eq. (32) it is easily seen that this leads to replacing δ with δ/m in Eq. (34).

VI. Failure Rate Proportional to System Stress

In this section we will consider three related models of hazard rate processes. In the first model we assume the rate of failure is proportional to the number of customers using the system, for example the rate of wear on a highway may have a component which is proportional to the number of automobiles, N(t), using the highway. In the second model the failure rate of a major system (for example on a battleship during combat) is increased proportional to the amount of un-repaired damage (the number of hits not repaired is $N(t)$, the amount of damage/hit is some positive r.v.). Both of these models are related to a generalized M/G/∞ queue. Finally, we model the reliability of electronic systems in a laser battle scenario. Failures may be caused by impulsive stress caused by a laser hit, or by accumulated heat from the laser hits. It is found that none of the models in this section lead to a constant effective hazard rate.

<u>Highway model</u>

Consider a large multi-lane highway in which the traffic level is such that cars do not interfere with each other. The number of cars on the highway can be modeled as an M/G/∞ queue. The rate of arrivals of cars to the highway is $\gamma(t)$ and the pdf of the time spent on the highway section of interest is $b(\tau)$ (the "service" distribution). The hazard rate is modelled as $h(t) = \lambda(t) + \beta N(t)$. We show in Appendix C that the characteristic functional of the hazard rate can be written as,

$$C_h[\eta(\cdot)] = \exp\{ -\int_0^t \lambda(t')\eta(t')dt'\} \cdot \exp\{ \int_0^t \gamma(t')dt'[1 - \int_0^\infty d\tau b(\tau)\exp[i\int_{t'}^{t'+\tau} \beta\eta(s)ds]\,] \}. \tag{35}$$

For example, when the transit time is exponentially distributed, parameter μ, the reliability can be evaluated as,

$$R(t) = e^{-\lambda t} \exp\{ -\gamma\beta t/(\mu+\beta) + \gamma\beta/(\mu+\beta)^2 [1 - e^{-(\mu+\beta)t}] \}. \tag{36}$$

And when the transit time is equal to τ with probability one, the reliability is,

$$R(t) = e^{-\lambda t} \exp\{ -\gamma t - \gamma/\beta \cdot [1 - e^{-\beta\tau}] - \gamma(t-\tau)e^{-\beta\tau}\}, \qquad (37)$$

when $t > \tau$, and

$$R(t) = e^{-\lambda t} \exp\{ -\gamma t - \gamma/\beta \cdot [1 - e^{-\beta t}] \}, \qquad (38)$$

when $t < \tau$. These formulas generalize as shown above for components in series and/or parallel systems.

Battle Damage Model

Assume that incoming rounds hit a ship according to a non-homogeneous Poisson process, rate $\gamma(t)$, during a battle. The amount of damage to the ship per round (or, really, the increased system stress resulting from each hit) is given by a random variable, β, with some non-negative distribution. Let the pdf of the time to repair the damage from each hit be $b(\tau)$. If the failure rate of the total system contains a term which increases proportional to the amount of un-repaired damage (possibly because of increased demands made on the rest of the system), then the hazard rate is,

$$h(t) = \lambda(t) + k \sum_{j=1}^{n} \beta_j \{\theta(t - t_j) - \theta(t - t_j - \tau_j)\}. \qquad (39)$$

The stochastic portion of this corresponds, for example, to the total weight of customers in a non-homogeneous $M/G/\infty$ queue, where each customer has a random weight, β_j. In Appendix C we derive the characteristic functional for this process. Applying that result we have, in general,

$$R(t) = e^{-\lambda t} \exp\{ -\int_0^t dt' \gamma(t') [1 - C_\beta(\int_{t'}^{t'+\tau} \eta(s)ds) b(\tau) d\tau]\}\Big|_{\eta(\cdot)=i} . \qquad (40)$$

where C_β is the ordinary characteristic function of β. If β is a constant this reduces to the previous model. Again, for m components in series replace λ with $m\lambda$ and evaluate the characteristic functional for $\eta(\cdot) = m \cdot i$. For parallel configurations Eq. (C4) can be used in conjunction with Eq. (31).

One example that can be worked out in closed form is when β is an exponential random variable, parameter ϵ, and the system cannot be repaired during the battle, i.e. $b(\tau) = \delta(\tau - \infty)$. The reliability of a single component is easily calculated using Eq. (40) to be,

$$R(t) = e^{-(\gamma+\lambda)t} \cdot \{ (\epsilon + t)/\epsilon \}^{\epsilon\gamma}, \qquad (41)$$

and for m identical components in series we obtain,

$$R(t) = e^{-(\gamma+m\lambda)t} \cdot \{ (\epsilon + mt)/\epsilon \}^{\epsilon\gamma/m}. \qquad (42)$$

Laser-Battle Model

We now model the failures of electronic systems in a laser battle as, 1) immediate failures resulting from impulsive thermal or kinetic shocks, or 2) random failures brought about by the accumulated thermal stress from repeated hits by laser beams. The background hazard rate is again represented by $\lambda(t)$. Assuming an exponential cooling law, the hazard rate resulting from laser hits at times $\{t_j\}$ is,

$$h(t) = \lambda(t) + \sum_{j=1}^{n} \{\alpha\, \delta(t-t_j) + \beta\, \theta(t - t_j)\, e^{-\kappa(t-t_j)}\}, \qquad (43)$$

where the second term represents the contribution to ΔT (increase in the system temperature) resulting from the laser strike at time t_j. Using the same methods as in the Appendices it can be shown that the characteristic functional for the stochastic part of this process is,

$$C[\eta(\cdot)] = \exp\{\int_0^t dt'\gamma(t')\, [\, \exp[\int_{t'}^t ds\, \eta(s)\, r(s - t')] - 1\,]\, \}, \qquad (44)$$

where,

$$r(s - t') = \alpha\, \delta(s-t') + \beta\, \theta(s - t')\, e^{-\kappa(s-t')}. \qquad (45)$$

Hence, the reliability is,

$$R(t) = e^{-\lambda t} \exp\{\int dt'\gamma(t')\, [\, \exp[-\alpha - \beta/\kappa\cdot(1 - e^{-\kappa(t-t')})] - 1\,]\, \}. \quad (46)$$

For m identical components in series, seeing exactly the same environment, Eq. (44) is modified by multiplying λ, α and β by m. The non-constant effective hazard rate, $\hat{h}(t)$, can be read directly from Eq. (46), e.g. for m identical components in series it is,

$$\hat{h}(t) = m\lambda(t) + \gamma(t)\{\exp[-m\alpha - m\beta/\kappa\cdot(1 - e^{-\kappa(t-t')})] - 1\}. \quad (46)$$

Note that γ is not multiplied by m. All components see exactly the same shocks from laser strikes.

More generally, we could model different types of laser weapons, ranges, atmospheric propagation, etc. by taking α and β as (correlated) random variables. The above equations can also be generalized if the components are not identical but have different vulnerabilities, thermal conductivities, etc.

VII. Conclusion

We have presented a general formalism for calculating the reliability of multi-component systems subject to a random environment. The correlation in failures of different components can be accounted for by calculating the system reliability in terms of appropriate combinations of the characteristic functional of the random hazard rate, evaluated for $\eta(\cdot) = i*j$ $\{j=1, n;\ i = \sqrt{(-1)}\}$. We have shown how the method may be used in different circumstances by applying it to a number of different models for the random environment. Our results include non-constant effective hazard rates for some of the random environment models. The intuitively reasonable result that the reliability of series (parallel) systems is greater (less) in a correlated environment than when the components see independent environments has been demonstrated for some of our models. Although we have only considered series or parallel systems, the application to more complicated systems containing components both in series and in parallel is straightforward.

Acknowledgement

The author is grateful to Prof. D. P. Gaver, Jr. for suggesting the possibility of applying the characteristic functional to reliability problems.

Footnotes

1. The notion of random variability of the environment is introduced in D. P. Gaver, Jr. "Random Hazard in Reliability Problems", Technometrics **5**, No. 2, pp. 211-226 (1963). The processes considered there have independent increments (e.g. the Gamma increments on p. 215), hence they yield a constant effective hazard rate and exponential lifetime distribution.

2. A. Mercer explores a model of wear-dependent failure rates that leads to a non-constant hazard rate in "Some Simple Wear-dependent Renewal Processes", J. Royal Stat. Soc. (B) **23**, pp. 368-376 (1961).

3. See for example, R. E. Barlow and F. Proschan, Statistical Theory of Reliability and Life Testing. (To Begin With Press, Silver Spring, MD) 1981.

4. M. S. Bartlett, An Introduction to Stochastic Processes, 2nd edition, (Cambridge University Press) 1966.

5. D. R. Cox and V. Isham, Point Processes, (Cambridge University Press) 1980.

6. E. Rockower, "The Gamma/Exponential Markov Process", NPS T.R. 1986

7. S. Ross, Stochastic Processes, (Wiley, New York) 1983, p. 212ff.

8. The idealized model of a textile yarn discussed in, D. R. Cox and H. D. Miller, The Theory of Stochastic Processes, (Wiley, New York) 1965, p. 366 ff., can be seen to be equivalent to the M/G/∞ queue.

Appendix A. Derivation of the Characteristic Functional for Impulses

We now derive the characteristic functional for the hazard rate resulting from random delta function impulses. Unless otherwise noted, we define the characteristic functional for the process over the interval 0 -> t. First, the random part of the hazard rate, g(t), is defined more explicitly as,

$$g(t') = \sum_{j=1}^{n} \alpha_j \, \delta(t' - t_j), \qquad \text{(A1)}$$

where n has a non-homogeneous Poisson distribution, p_n, with mean,

$$\overline{n} = \int_0^t \gamma(t)dt, \qquad \text{(A2)}$$

the t_j have pdf given by $\gamma(t')/\overline{n}$ (as is appropriate for a non-homogeneous Poisson process), and α_j have an arbitrary distribution, $f(\alpha_j)$, with ordinary characteristic function, C_α. The characteristic functional for g(t') is,

$$C_g[\eta(\cdot)] \quad = \quad E\{ \exp[i\int_0^t \eta(t)\, g(t)\, dt] \}, \tag{A3}$$

or,

$$= \quad \sum_{n=0}^{\infty} p_n \{ \prod_{j=1}^{n} [\int\int dt_j\, \gamma(t_j)/\overline{n}\; d\alpha_j\, f(\alpha_j)\, e^{i\, \alpha_j\, \eta(t_j)}] \},$$

where the integration over t_j is 0 -> t the integration over α_j is 0 -> ∞. We have made use of the Dirac delta function to perform the integration over t'. The product over j now reduces simply to the expression in the square brackets raised to the n^{th} power because of the independence of each term in the shot noise-like process. Performing the average over n (yielding the standard result for the generating function of a Poisson distribution) and the integration over α_j (yielding the characteristic function for the amplitude distribution) and taking the limit T --> ∞, results in,

$$C_{h,t}[\eta(\cdot)] \;=\; \exp\{ i\lambda_0\int_0^t \eta(t')dt' + \int_0^t \gamma(t')\, [C_\alpha(\, \eta(t')\,) - 1]dt'\}, \tag{A4}$$

where we have included the (independent) characteristic functional for the deterministic portion of the hazard rate.

Appendix B. The Characteristic Functional of the Gamma Process[6]

Using the results of Appendix A, Eq. (A4), the characteristic functional of random delta function impulses (rate $\upsilon\gamma(t)$) with exponential amplitudes (parameter δ), is,

$$C_g[\eta(\cdot)] = \exp\{ i \int_{-\infty}^{\infty} dt\, \upsilon\gamma(t)\, \eta(t) / [\delta - i\eta(t)] \}. \qquad (B1)$$

If this shot noise is fed into a first-order linear filter (decay parameter $\gamma(t)$), the resulting process, x, satisfies,

$$\dot{x} + \gamma x = g. \qquad (B2)$$

Now, to find the characteristic functional for the process, x(t), subject to the initial condition, $x(0) = x_0$, use the solution of the stochastic differential equation, (letting γ be constant to make it easier to follow the derivation) in the definition,

$$C_x[\eta(\cdot)] = E\{ \exp[i\int_0^{\infty} \eta(t)\, x(t)\, dt] \}, \qquad (B3)$$

or,

$$= \exp[ix_0 \int dt\, \eta(t) e^{-\gamma t}] \cdot \exp\{i \int dt\, \eta(t) e^{-\gamma t} \int dt' e^{\gamma t'}\, \theta(t-t')\, g(t') \},$$

where we have used the properties of the Heaviside unit step function, $\theta(t-t')$, so that all the integrations are over 0 -> ∞, thus making it easier to interchange the order of integration to give,

$$C_x[\eta(\cdot)] = \qquad \text{(B4)}$$

$$\exp[ix_0\int dt\ \eta(t)e^{-\gamma t}] \cdot \exp\{i\int dt'\ g(t')\ [\int dt e^{-\gamma(t-t')}\ \theta(t-t')\ \eta(t)]\ \}.$$

The second exponential term is now of exactly the same form as the characteristic functional for g(t), with the expression in square brackets replacing $\eta(t')$. Hence, making use of Eq. (B1) we have,

$$C_x[\eta(\cdot)] = \exp[ix_0\int_0^\infty dt\ \eta(t)e^{-\gamma t}] * \qquad \text{(B5)}$$

$$\exp\{i\int_0^\infty dt'\ \upsilon\gamma\ [\int_{t'}^\infty dt\ e^{-\gamma(t-t')}\ \eta(t)]\ /\ [\delta - i\ [\int_{t'}^\infty dt e^{-\gamma(t-t')}\eta(t)]\ ]\ \},$$

where we have used the properties of the step function.

To determine the marginal distribution of this process use $\eta(t) = \eta_0\ \delta(t-t_0)$, which recovers the ordinary characteristic function for $x(t_0)$. A straightforward calculation yields,

$$C_{x(t_0)}[\eta_0] = \exp[i\eta_0 x_0 e^{-\gamma t_0}] \cdot \{\ [\delta - i\eta_0\ e^{-\gamma t_0}]\ /\ [\delta - i\eta_0]\ \}^{\upsilon}. \qquad \text{(B6)}$$

for $\nu=1$, This is the characteristic function of a r.v. that, with probability $\alpha = e^{-\gamma t}$, has the value $x_0 e^{-\gamma t}$, and with probability $(1-\alpha)$ is the sum of $x_0 e^{-\gamma t}$ and a r.v. with exponential distribution of parameter δ.

Now, taking $t_0 \to \infty$ Eq. (B6) yields the marginal distribution of the steady state process,

$$C_{X(\infty)}[\eta_0] = \{\delta / [\delta - i\eta_0]\}^{\upsilon}, \tag{B7}$$

i.e. the characteristic function of a Gamma distribution, as promised.

Appendix C. The Characteristic Functional of an M/G/∞ Queue, and Extensions

The M/G/∞ queue may be modeled as shot noise impulses of unit magnitude[8] ("customers" arrive with rate $\gamma(t)$, n is again Poisson),

$$g(t) = \sum_{j=1}^{n} \delta(t - t_j), \tag{C1}$$

which have been filtered through a linear system with a random response function given by,

$$r(t - t_j) = \theta(t - t_j) - \theta(t - t_j - \tau_j), \tag{C2}$$

i.e. a unit height pulse with random duration, τ_j (the random "service" time). Hence, for $\beta_j = 1$, the number in the system at time t is,

$$N(t) = \sum_{j=1}^{n} \beta_j \{\theta(t - t_j) - \theta(t - t_j - \tau_j)\}. \tag{C3}$$

We have included the factor, β_j, so that we can allow the system response to also have a random height, corresponding to a random weight (or damage, system stress, pollution, etc.) of each customer. Using the definition of the characteristic functional and following the same steps as in the previous Appendices, we obtain,

$$\sum_{n=0}^{\infty} p_n \{ \prod_{j=1}^{n} [\int\int\int_{o}^{\infty} dt_j \gamma(t_j)/\bar{n}\, d\beta_j\, f(\beta_j)\, d\tau_j b(\tau_j) \exp[i\beta_j \int ds \eta(s)\, \Theta(s,t_j,\tau_j)] \},$$

where $\Theta(s,t_j,\tau_j) = \theta(t - t_j) - \theta(t - t_j - \tau_j)$. Again using the independence of each term in the product, and the fact that they are all the same, we can perform the summation over p_n (Poisson), and the average over β (obtaining the ordinary characteristic function, C_β). Finally, the characteristic functional may be written,

$$C[\eta(\cdot)] = \exp\{ -\int_{o}^{t} dt' \gamma(t')\, [1 - \int C_\beta(\int_{t'}^{t'+\tau} \eta(s) ds) b(\tau) d\tau \,] \}. \qquad \text{(C4)}$$