Some Results for the M/G/∞ Queue

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ABSTRACT

In this paper the characteristic functional is derived for the number of customers in a nonhomogeneous $M/G/\infty$ Queue for various initial conditions. Some possible applications are suggested. Comparison is also made with the characteristic functionals for the arrival and departure processes.

Some Results for the M/G/∞ Queue

1. Introduction & Summary

In this paper we will consider the following diverse, but mathematically similar phenomena:

A) The number of airplane Line Replaceable Units (LRUs) in the base and/or depot repair pipelines

B) The number of customers in a totally self-service facility

C) The number of particles within a given volume as a function of time when these particles either undergo Brownian motion or some other type of migration.

The results we obtain will be general enough to encompass not only the transient regime before any steady state is reached, but also the nonstationary case for which no steady state may exist. The mathematical technique which we employ to arrive at some of these results is the characteristic functional. Although it is somewhat novel in Queueing Theory, the related generating functional has been utilized effectively by Vere-Jones (1968) in the study of arrival and departure processes.

We will derive the characteristic functional for the number of customers in an $M/G/\infty$ queue. The arrival process is assumed to be a non-homogeneous Poisson process with arrival rate $\lambda(t)$, a general positive function of time. We denote this $M(t)/G/\infty$. The stochastic process $N(t)$ is not assumed to be in any state of equilibrium, were it to exist. We treat the transient case in which, a) $N(o) = m$, or b) $N(o)$ has a Poisson distribution with mean m_o.

The ordinary characteristic function is a convenient tool for dealing with a single, finite or a denumerable number of random variables. A generalization of this is the characteristic functional which can be defined for an arbitrary random variable $X(t)$ (Bartlett, 1955) . We define it as the following expectation, possibly conditional on some initial constraint, over all possible realizations (or "paths") of $X(t)$ and weighted with the appropriate probability "density" of each path:

$$C[\eta(\cdot)] = E\left\{e^{i\int \eta(t)\, X(t)\, dt}\right\}$$

The functions $\eta(t)$ are arbitrary, suitably well behaved test functions. Various correlation functions may be obtained by taking multiple functional derivatives of $C[\eta(\cdot)]$.

For example:

$$\frac{1}{i^2}\frac{\delta}{\delta\eta(t)}\frac{\delta}{\delta\eta(t')}C[\eta(\cdot)]\Big|_{\eta=0} = E\{X(t)\,X(t')\}$$

2. Derivation of the Characteristic Functional

We now derive the characteristic functional of N(t), the number in the system at time t $\geq$ o. It is assumed that there are m_o customers in the system at t = o.

Imagine that the total population of possible customers is N + m_o of which m_o are in the queue at t = o. Define a function ψ_i (t) (KAC 1959) , i = 1,2··N + m_o, which is zero if customer i is not in the system at time t and which is one if he is in the system. Then the total number of customers in the system at any time t is:

$$N(t) = \sum_{i=1}^{N+m_o}\psi_i(t) = \sum_{i=1}^{N}\psi_i(t) + \sum_{i=N+1}^{N+m_o}\psi_i(t) \qquad (1)$$

$$= N_{new}(t) + N_{old}(t)$$

The initial conditions at t = o are:

$$N_{new}(0) = 0 \quad \Leftrightarrow \quad \psi_i(0) = 0\ ;\ i = 1,2\cdots N$$

$$N_{old}(0) = m_o \quad \Leftrightarrow \quad \psi_i(0) = 1\ ;\ i = N+1,\cdots N+m_o$$

We now assume the N customers will definitely arrive at the queue sometime between 0 and T. N and T will later go to infinity. If the arrival process were stationary we could set $\lambda = N/T$. However, λ is a function of time; the probability that any "new" customer will arrive at time t is proportional to $\lambda(t)$. Normalizing to one the distribution between $0 \rightarrow T$ for each customer, i, we have the probability of the new customer, i, arriving in dt; $i \in [1, N]$:

$$\frac{\lambda(t)dt}{\int_0^T \lambda(t)dt}$$

Since the probability that customer i will arrive between 0 and T is certain this yields one when integrated from 0 to T.

Also the total number of arrivals between 0 and T is N. Therefore, since the overall arrival rate is $\lambda(t)$:

$$\int_0^T \lambda(t)\,dt = N \tag{2}$$

Consequently, the probability that "new" customer i is in the system at an arbitrary point t between 0 and T is:

$$\begin{aligned} P_r\left[\psi_i(t) = 1 \mid \psi_i(0) = 0\right] &= E\{\psi_i(t)\}_{\psi_i(0)=0} \\ &= \int_0^t P_r\left[\text{service time} \geq t-x \mid \text{"}i\text{" arrives at } x\right] P_r\left[\text{"}i\text{" arrives at } x\right] dx \end{aligned}$$

Using the above results $\big(B(t)$ is the service CDF$\big)$, we obtain:

$$E\{\psi(t)\}_{\psi(0)=0} = \int_0^t [1-B(t-x)] \frac{\lambda(x)}{N} \, dx \tag{3}$$

Now considering the "old" customers, present at t = o, and using a result from renewal theory for the distribution of remaining service time (this is equivalent to assuming that the old customers were in a steady state condition),

$$P_r[\psi(t)=1 \mid \psi(0)=1] = E\{\psi(t)\}_{\psi(0)=1} = \int_t^\infty \frac{[1-B(x)]}{1/\mu} \, dx \tag{4}$$

By the independence of the different customers and servers in the M/G/∞ queue;

$$C[\eta(\cdot)] = E\left\{e^{i\int \eta(t) N(t)\, dt}\right\}$$

$$= E\left\{e^{i\int \eta(t) N_{new}(t)\, dt}\right\} E\left\{e^{i\int \eta(t) N_{old}(t)\, dt}\right\}$$

Again using this independence and Equation (1).

$$C[\eta(\cdot)] = E\left\{e^{i\int \eta(t)\psi(t)\, dt}\right\}^N_{\psi(0)=0} E\left\{e^{i\int \eta(t)\psi(t)\, dt}\right\}^{m_0}_{\psi(0)=1} \tag{5}$$

We now consider the first part of Equation (5), the "new" customers, by expanding the exponential:

$$E\left\{e^{i\int \eta(t)\psi(t)\,dt}\right\}_{\psi(0)=0} = 1 + i\int \eta(t)\, E\{\psi(t)\}_{\psi(0)=0}\, dt$$

$$+ \sum_{n=2}^{\infty} \frac{(i)^n}{n!} \iint\cdots\int dt_1 \cdots dt_n\, \eta(t_1)\cdots\eta(t_n)\, E\{\psi(t_1)\cdots\psi(t_n)\}_{\psi(0)=0}$$

Now making use of the fact that if a customer is in the system at t_1 and t_2 he is also in the system at all intermediate times, and the fact that $\psi = \left.\begin{matrix}0\\1\end{matrix}\right\}$, we have:

$$E\{\psi(t_1)\cdots\psi(t_n)\}_{\psi(0)=0} = E\{\psi(t_<)\,\psi(t_>)\}_{\psi(0)=0}$$

where $t_<$ and $t_>$ are the minimum and the maximum of the set $\{t_i\}_1^n$.

Also:

$$E\{\psi(t_<)\,\psi(t_>)\}_{\psi(0)=0} = P_r\left[\text{in service at } t_< \text{ and } t_> \mid \text{not in service at } 0\right]$$

$$= \int_0^{t_<} P_r\left[\text{in service at } t_> \mid \text{arrived at } x \in [0, t_<]\right] P_r\left[\text{arrive at } x\right] dx$$

$$= \int_0^{t_<} \left[1 - B(t_> - x)\right] \frac{\lambda(x)}{N}\, dx$$

After inserting a factor of n(n-1) to allow for the number of different ways in which the maximum and minimum can be chosen from $t_1 \cdots t_n$, the "sum over n" term above becomes,

$$\sum_{n=2}^{\infty} \frac{(i)^n}{n!} n(n-1) \int_0^{\infty}\int_0^{t_>} \left[\int_{t_<}^{t_>} \eta(t)\,dt\right]^{n-2} \int_0^{t_<} [1-B(t_>-x)] \frac{\lambda(x)}{N} dx\, \eta(t_<)\eta(t_>)\, dt_< dt_> \tag{6}$$

Now use (3) for the term 1st order in η and use (6):

$$E\left\{e^{i\int \eta(t) N_{new}(t)\,dt}\right\} = \lim_{N\to\infty}\left[1 + i\int_0^{\infty}\eta(t)\int_0^t [1-B(t-x)]\frac{\lambda(x)}{N}dx\,dt + \frac{1}{N}\sum_{n=2}^{\infty}\frac{(i)^n}{(n-2)!}\iint(\cdots)\right]^N$$

Now take the *limit* $N \to \infty$ (and $T \to \infty$) and use $\lim_{N\to\infty}(1+\frac{x}{N})^N = e^x$;

$$E\left\{e^{i\int \eta(t) N_{new}(t)\,dt}\right\} = \exp\left[i\int_0^{\infty}\eta(t)\int_0^t [1-B(t-x)]\lambda(x)\,dx\,dt + \sum_{n=2}^{\infty}\frac{(i)^n}{(n-2)!}\int_0^{\infty}\int_0^{t_>}\left[\int_{t_<}^{t_>}\eta(t)\,dt\right]^{n-2}\int_0^{t_<}[1-B(t_>-x)]\lambda(x)\,dx\,\eta(t_<)\eta(t_>)\,dt_<dt_>\right] \tag{7}$$

The infinite series in the exponent can be summed up and we will do this later. The present form is, however, convenient for taking functional derivatives.

We now consider the term corresponding to the customers already in service at t = o.

$$E\left\{e^{i\int\eta(t)\psi(t)dt}\right\}_{\psi(0)=1} = 1+i\int\eta(t)\,E\{\psi(t)\}_{\psi(0)=1}\,dt$$

$$+\sum_{n=2}^{\infty}\frac{(i)^n}{n!}\iint\cdots\int\eta(t_1)\cdots\eta(t_n)\,E\{\psi(t_1)\cdots\psi(t_n)\}_{\psi(0)=1}\,dt_1\cdots dt_n \qquad (8)$$

Now note that:

$$E\{\psi(t_1)\cdots\psi(t_n)\}_{\psi(0)=1} = E\{\psi(t_>)\}_{\psi(0)=1}$$

To allow for the number of ways in which the maximum ($t_>$) can be chosen we insert a factor of n in the sum.

Hence (8) becomes:

$$1+i\int_0^{\infty}\eta(t_>)E\{\psi(t_>)\}_{\psi(0)=1}\,e^{i\int_0^{t_>}\eta(t)dt}\,dt_> \qquad (9)$$

And the characteristic functional is obtained by raising (9) to the mth power and multiplying by (7) and using (4) for $E\{\Psi(t_>)\}_{\Psi(0)=1}$;

$$C[\eta(\cdot)] = \exp\Big[i\int_0^\infty \eta(t)\int_0^t [1-B(t-x)]\lambda(x)\,dx\,dt$$

$$+ \sum_{n=2}^\infty \frac{(i)^n}{(n-2)!}\int_0^\infty\int_0^{t_>}\Big[\int_{t_<}^{t_>}\eta(t)\,dt\Big]\int_0^{t_<}[1-B(t_>-x)]\lambda(x)\,dx\,\eta(t_<)\eta(t_>)\,dt_<\,dt_>\Big]\cdot \tag{10}$$

$$\cdot\left\{1+i\int_0^\infty \eta(t_>)\int_{t_>}^\infty \frac{[1-B(x)]}{1/\mu}\,dx\, e^{i\int_0^{t_>}\eta(t)\,dt}\,dt_>\right\}^m$$

With sufficient patience one can now calculate all multiple time correlation functions of the form:

$$E\{N(t_1)N(t_2)\cdots N(t_n)\}_{N(0)=m} = \frac{1}{i^n}\frac{\delta}{\delta\eta(t_1)}\cdots\frac{\delta}{\delta\eta(t_n)}C[\eta(\cdot)]\Big/_{\eta=0}$$

We now consider the case in which the number of customers in the system at $t = o$ is not certain, but has a Poisson distribution with mean $\bar{m}_o$. To obtain the characteristic functional for this case we average the functional obtained above, which we now denote $C_m[\eta(\cdot)]$, over a Poisson distribution:

$$C_{\bar{m}_o}[\eta(\cdot)] = \sum_{m=0}^\infty C_m[\eta(\cdot)]\frac{\bar{m}_o^{\,m}e^{-\bar{m}_o}}{m!}$$

This summation is easily accomplished. After changing variables and performing a number of integrations by parts, the summations in equation (10) are also accomplished and we have:

$$C_{\bar{m}_0}[\eta(\cdot)] = \exp\left\{\int_0^\infty \left[\int_0^\infty \frac{dB(s)}{ds} e^{i\int_{t_<}^{t_<+s}\eta(t')dt'} ds - 1\right]\lambda(t_<)dt_< + m_0\int_0^\infty \frac{[1-B(t_>)]}{1/\mu}\left[e^{i\int_0^{t_>}\eta(t')dt'} - 1\right]dt_>\right\} \tag{11}$$

3. Applications of the Characteristic Functional

We now make two observations:

1) (Renewal Processes) The expression (9) raised to the mth power is the characteristic functional for the number of machines working as a function of time, after the repairmen go on strike. Or, it is the characteristic functional for the number of light bulbs which are still working, out of m, after we have run out of replacement bulbs.

2) The characteristic functional, equation (11) is of a form eminently suited to calculating the cumulants K_n (Theile semi-invariants, linked moments) defined by:

$$E\left\{e^{i\int\eta(t)N(t)dt}\right\} = \exp\left[\sum_1^\infty \frac{(i)^n}{n!}\int\int\cdots\int dt_1\cdots dt_n\,\eta(t_1)\cdots\eta(t_n)K_n(t_1\cdots t_n)\right]$$

To obtain the ordinary characteristic function for $P_n(t)$ for a nonstationary arrival process with rate $\lambda(t)$ and a Poisson distribution in the queue at t = o we set $\eta(t') = \eta \cdot \delta(t'-t)$.

Substituting this in equation (11) we obtain after some calculation:

$$C_{\bar{m}_o}(\eta, t) = \exp\left\{ (e^{i\eta} - 1)\, \bar{n}(t) \right\} \tag{12}$$

where:

$$\bar{n}(t) = \int_{-\infty}^{t} [1 - B(t-x)]\, \tilde{\lambda}(x)\, dx \tag{13}$$

Here the extended definition of $\tilde{\lambda}$ is:

$$\tilde{\lambda}(x) = \begin{cases} \lambda(x) & ; \; x > o \\ \bar{m}_o \mu & ; \; x \leq o \end{cases}$$

Equation (12) is seen to be the ordinary characteristic function of a Poisson distribution with mean $\bar{n}(t)$.

For the stationary case where λ = constant and $\bar{m}_o$ = o for $M/G/\infty$, this is seen to yield the standard result on p.273 in Gross and Harris (1974). Also for the $M(t)/M/\infty$ case the $P_n(t)$ obtained for $\lambda(t)$ a function of time can be shown to satisfy the appropriate $M/M/\infty$ Kolmogorov eq'n.

The case we have been considering is seen to correspond to λ equal to a constant ($\bar{m}_o \mu$) for $t < o$. Alternatively we could have considered a more general $\lambda(t)$, for all t, and Equations (12) and (13) would still be valid.

Multi-Time Correlation Functions of N(t)

As indicated above the characteristic functional for $\lambda(x)$ a general positive function ($C_N[\eta(t)]$ not conditional on the distribution at time o) can be obtained by extending the lower limit from o to $-\infty$ in the integrals.

The first order cumulant is the coefficient of $i\eta(t)$:

$$K_1(t) = \int_{-\infty}^{t} [1 - B(t-x)]\lambda(x)\,dx = \bar{n}(t)$$

This agrees with what we obtained above for $\bar{n}(t)$.

The second order cumulant (and correlation function) is :

$$K_2(t_1,t_2) = \frac{1}{i^2}\frac{\delta}{\delta\eta(t_1)}\frac{\delta}{\delta\eta(t_2)} \ln C_N(\eta)$$

$$= \int_{-\infty}^{t_1} [1-B(t_2-x)]\lambda(x)\,dx\ \theta(t_1-t_2)$$

$$+ \int_{-\infty}^{t_2} [1-B(t_1-x)]\lambda(x)\,dx\ \theta(t_2-t_1)$$

The unit step function is denoted by $\Theta(x)$.

The correlation functions can thus be obtained either by functional differentiation of $C[\eta(\cdot)]$ or by using the standard relations between the cumulants and the correlation functions.

4. Special Choices of $\eta(t)$ (A Further Application of $C_N[\eta(\cdot)]$)

By choosing a particular test function of $\eta(t') = \eta \cdot \delta(t'-t)$ above, we obtained the ordinary characteristic function of the number in the $M(t)/G/\infty$ Queue at one time t. Similarly, by choosing $\eta(t') = \eta_1 \delta(t'-t_1) + \eta_2 \delta(t'-t_2)$ we would obtain the characteristic function for the joint process consisting of the number in the Queue at t_1 and at t_2. For the case in which λ is a constant this is seen to be equivalent to the Smoluchowski process of the fluctuating number of particles in a given region (cf. Kac 1959). The latter has been applied to the fluctuating number of particles in a region undergoing Brownian motion. The Smoluchowski theory has also been applied by Fürth to estimating the speed of pedestrians on a sidewalk by successive observation of the number within a given region and evaluating the "probability after effect" (Kac 59, Chandrasekhar 1943). Conversely, our result for non-constant λ may be viewed as a generalization of the Smoluchowski process for cases in which the concentration of particles in the fluid is changing in time.

By setting $\eta(t) = \begin{cases} \eta/(t_2 - t_1) & ; \; t_1 < t < t_2 \\ 0 & ; \; \text{elsewhere} \end{cases}$

we obtain the characteristic function for the time averaged number of customers in the Queue between t_1 and t_2.

A particular case which may be of interest is that in which $t_1 = o$ and $N(o) = o$. This would then represent the time averaged number of airplane Line Replaceable Units (LRUs) in the base and/or depot repair pipeline for a time period just after the base has begun operating. The expected value of this time average is easily obtained from our Equation (13) for $\bar{n}(t)$. However, the variance about this mean may be helpful in determining a safe level of initial spares required at the base.

An easily calculated example is a time homogeneous $M/M/\infty$ Queue. The characteristic function for this time averaged N(t) is:

$$C(\eta) = \exp\left\{ \frac{\lambda t_0 \left(e^{i\eta - \mu t_0} - 1\right) i\eta}{(i\eta - \mu t_0)^2} - \frac{i\lambda t_0 \eta}{i\eta - \mu t_0} \right\}$$

The expectation of the time averaged N(t) is:

$$\frac{1}{i}\frac{d}{d\eta} C(\eta)\Big|_{\eta=0} = \frac{\lambda}{\mu} + \frac{\lambda}{\mu^2 t_0}\left[e^{-\mu t_0} - 1\right]$$

This could also have been obtained from $\bar{n}(t)$. The variance is:

$$\frac{1}{i^2}\left(\frac{d}{d\eta}\right)^2 \ln C(\eta)\Big|_{\eta=0} = \frac{2\lambda}{t_0\mu^2}\left(1+e^{-\mu t_0}\right)$$

5. <u>Estimate of the Number in Queue After λ =o</u>

We now prove that the a posteriori distribution of the number of customers in the system at a time t after λ falls to zero is independent of the number of customers observed to have departed during the time period from o to t. λ is assumed to be constant up to zero time and zero afterwards. We have from (12) and (13):

$$P(n,t) = \frac{[\bar{n}(t)]^n e^{-\bar{n}(t)}}{n!}$$

where for the case we are considering the upper limit on the integrals can be set to zero for times greater than zero. At t=o we have;

$$P(n,0) = \frac{[\bar{n}_0]^n e^{-\bar{n}_0}}{n!}$$

where

$$\bar{n}_0 = \lambda \int_{-\infty}^{0} [1-B(-x)]\, dx = \frac{\lambda}{\mu}$$

by the usual definition of μ.

We now calculate the a posteriori distribution of the number in the Queue given that we have observed m customers to have left between o and t. The "a priori" distribution is the above p(n,o) which is to be estimated and updated by consideration of the departures.

The conditional probability we seek is:

$$p(n,t|m) \equiv P_r[n \text{ customers remaining at } t, \text{ given that } m \text{ left}]$$

$$p(n,t|m) = \frac{p(n,t;m)}{\tilde{p}(m,t)}$$

where: $\tilde{p}(m,t) = P_r[m \text{ departures between } o \text{ and } t]$

The joint probability p(n,t;m,); if each customer in the Queue at time has probability p of leaving before t, is:

$$p(n,t;m) = P_r[m \text{ departures} \mid m+n \text{ customers at } o] \cdot P_r[n+m \text{ in } Q \text{ at } o]$$

$$= \left[\binom{n+m}{m} p^m (1-p)^n\right] \cdot \frac{\bar{n}_o^{(n+m)} e^{-\bar{n}_o}}{(n+m)!}$$

From renewal theory we have:

$$p = \int_o^t \frac{[1-B(x)]\,dx}{1/\mu}$$

Now the probability of m departures, $\tilde{p}(m,t)$ is:

$$\tilde{p}(m,t) = \sum_{n=0}^{\infty} \binom{n+m}{m} p^m (1-p)^n \frac{\bar{n}_0^{(n+m)} e^{-\bar{n}_0}}{(n+m)!}$$

$$= \frac{(p\bar{n}_0)^m e^{-p\bar{n}_0}}{m!}$$

Hence:

$$p(n,t/m \text{ have departed}) = \frac{\binom{n+m}{m} p^m (1-p)^n \frac{\bar{n}_0^{(n+m)} e^{-\bar{n}_0}}{(n+m)!}}{(p\bar{n}_0)^m e^{-p\bar{n}_0}/m!}$$

And hence:

$$p(n,t/m \text{ have departed}) = \frac{[(1-p)\bar{n}_0]^n e^{-(1-p)\bar{n}_0}}{n!}$$

Since this is <u>independent</u> of the number observed to have left we have proven our assertion. Therefore, no further information is obtained concerning the remaining number of customers by observing the departures!

Also, by the definition of p given above, it is easily shown that this a posteriori distribution is equal to the a priori distribution given above, i.e, p(n,t).

$$(1-p)\bar{n}_0 = \left[1-\int_0^t \frac{[1-B(x)]}{1/\mu}dx\right]\bar{n}_0$$

and since $\bar{n}_0 = \lambda/\mu$

$$(1-p)\bar{n}_0 = \lambda\int_{-\infty}^{0}[1-B(t-x)]dx \equiv \bar{n}(t)$$

This is also required by consistency. So we have:

$$p(n,t/m \text{ departures}) = p(n,t)$$

This result complements Mirasol's (1963) paper concerning the output of an $M/G/\infty$ Queue. Our methods may be used to derive the generalization of Mirasol's results for non-homogeneous arrival and service rates.

6. Characteristic Functionals for the Arrival and Departure Processes

By methods similar to the ones used above one can derive the characteristic functionals for the number of customers, X(t) who have arrived before t and for the number of departures, $Y(t)$, from the Queue. We present the result here along with $C_N[\eta(\cdot)]$ for comparison. We give the case for which $\lambda = o$ for $t < o$.

$$C_x[\eta(\cdot)] = \exp\left\{\int_0^\infty \lambda(y)\left(e^{i\int_y^\infty \eta(t')dt'} - 1\right) dy\right\}$$

$$C_y[\eta(\cdot)] = \exp\left\{\int_0^\infty \lambda^D(y)\left(e^{i\int_y^\infty \eta(t')dt'} - 1\right) dy\right\}$$

$$C_N[\eta(\cdot)] = \exp\left\{\int_0^\infty \left[\int_0^\infty \frac{dB(s)}{ds} e^{i\int_y^{y+s} \eta(t')dt'} - 1\right]\lambda(y)dy\right\}$$

$$X(t) = N(t) + Y(t)$$

In $C_y[\eta(\cdot)]$ the effective departure rate is:

$$\lambda^D(t) = \int_0^t \frac{dB(x)}{dx}\lambda(t-x)dx$$

$C_x[\eta(\cdot)]$ and $C_y[\eta(\cdot)]$ are seen to be of the same form. Hence the departure process is nonhomogeneous Poisson (cf. Mirasol 1963 and Newell 1966).

Many possible applications of the characteristic functional method remain to be explored further. However, it is suggested that interesting and useful results may reward further efforts along these lines, not only for the nonstationary $M/G/\infty$ but perhaps for other Queues as well.

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