Evolution of the quantum statistics of light

Edward B. Rockower
Ketron, Incorporated, Valley Forge Executive Mall, Building 10, 530 East Swedesford Road, Wayne, Pennsylvania 19087

N. B. Abraham* and S. R. Smith
Department of Physics, Bryn Mawr College, Bryn Mawr, Pennsylvania 19010

(Received 29 August 1977)

Recently, the question of the origin of correlations in thermal light has again been debated. The alternative viewpoints suggest that (i) stimulated emission causes the correlations implied by Bose-Einstein statistics and exemplified by the Hanbury Brown–Twiss effect, or (ii) stimulated emission coherently amplifies any initial field, with no change in the nature of photon statistics, apart from the addition of, and interference with, spontaneous-emission noise. It is shown that two essentially different physical models have been considered previously, with many results being only approximate. For these models we obtain the evolution of the photon statistics by deriving exactly the generating function for the diagonal elements of the field density matrix and the quantum characteristic function. The source of Bose-Einstein correlations is seen to be merely the interference in the superposition of random independent fields from a chaotic source, spontaneous emission being one example. Extending the investigation to a nonlinear model, we show that, although there is no longer coherent amplification, it is still not correct to infer that this interaction, which produces the amplification, also leads to Bose-Einstein statistics.

I. INTRODUCTION AND SUMMARY

In a recent series of papers, the origin of Bose-Einstein (BE) correlations has been ascribed either to the stimulated emission inherent in the generation of light from a thermal source[1-6] or to the random superposition of the spontaneous emissions which is then coherently amplified by the action of stimulated emission.[7-12] According to the former viewpoint, any additional photon correlations arising from subsequent amplification of this chaotic light (or any other, possibly coherent, light) are just what one should expect from the action of the stimulated emission process. However, according to the second viewpoint, the added photon correlations result from the interference of the coherently amplified input field with the Gaussian spontaneous emission noise which is inherent in the light amplification process.

A laser amplifier operating far below the lasing threshold, so that saturation effects are not important, is well known to be describable as a linear amplifier plus a Gaussian noise source.[7-9,11,13,14] Whether this is consistent with the discrete emission of photons by atoms and the quantum nature of the field is at issue. In an experiment[15] designed to detect any deviations from linear amplification of the spontaneously generated light, Scarl and Smith found no excess photoelectron pair correlations in a detector illuminated by a He-Ne discharge tube apart from those to be expected from the ordinary Hanbury Brown–Twiss effect[16] for linearly amplified spontaneous emission noise. However, this result has also been interpreted as being consistent with the first mentioned viewpoint[1] in that the additional correlations from the amplification process have been said to result from those stimulated emissions which are also responsible for the amplification.

Given that the results of experiment (and of some calculations) are subject to interpretation, one tries to determine, theoretically, whether or not stimulated emission amplifies coherently. Although stimulated and spontaneous emission are fundamentally inseparable processes in a second-quantized theory, one seeks to know if in certain situations their effects are separable and if so how they serve as sources for the observed correlations.

Several approaches have been pursued in the theoretical investigation of this problem, with the proponents of each one apparently feeling that the issue was settled. A number of investigators have identified certain terms in (i) the equation for the time evolution of the radiation field density matrix in the number representation,[1,2,6,7] (ii) Einstein's derivation of blackbody radiation,[3-6] or (iii) the Fokker-Planck equation in the P representation,[7,17] as corresponding to stimulated or to spontaneous emission processes. Although recognizing that such identifications may be largely of pedagogical value, and may be valid only to first order of perturbation theory, they draw conclusions about the effects of these terms on the evolution of the photon statistics. While some of these investigations of the evolution of the field fluctuations were only approximate (perturbation) solutions of partial properties (first and second moments only) of the distribution, we will solve, exactly, for the photon number generating functions for the proposed spontaneous emission, stimulated emission, and combined model equations.

Another approach has been to investigate the properties of the quantum characteristic function which we define as[18]

$$C(\zeta,\eta) = \mathrm{tr}[\rho \exp(\zeta a^\dagger) \exp(\eta a)] \ . \qquad (1)$$

By noting that the characteristic function describing the field produced by an unsaturated amplifier factors into independent components,[19] including a linear amplification term and an amplified Gaussian noise term, the effects of linear amplification and additive noise have been attributed to stimulated and spontaneous emission, respectively.[9,11,13] In no case has the source of these effects been identified in the basic operator equations. We will show that the separate effects have their origin in particular terms in the operator time evolution equation of the density matrix, and we will justify the association of these terms (and their effects) with stimulated and spontaneous emission. We will also derive the characteristic function more simply and with a method which is perhaps more easily generalized to multiphoton processes.

In addition to the different mathematical methods one can use and the problems associated with trying to separate spontaneous and stimulated effects, we suggest that part of the conceptual difficulty that has been experienced has resulted from the existence of at least two, essentially different, physical models. A laser-amplifier model of a large number of atoms coupling to a mode of the electromagnetic field in a cavity and with a constant population of the upper and lower atomic levels may plausibly lend itself to the idea of linear and coherent amplification of an impressed field. This model has been considered by authors of both viewpoints.[2,7,17] On the other hand, the possibility of excess photon correlations arising from stimulated emission seems more compatible with the idea of a single atom,[1] initially in an excited state, interacting with an incident electromagnetic field and possibly making a transition to the ground state after a spontaneous or stimulated emission.[20] It should not be surprising that, as we will show, this model leads to conclusions different from theories involving linear amplification plus noise, but this is true only because of the nonlinear nature of the exact solution. However, to lowest order in time, the single-atom model can be interpreted in the same manner as was the laser-amplifier model.[21]

To aid us in understanding the role played by the physical model, we extend our considerations to include the following systems interacting with a single mode of the radiation field: (a) the exact, single, isolated atom (ESA); (b) the single, isolated atom (SA) to lowest order in time; (c) the unsaturated laser (UL) amplifier, a many-atom case corresponding to SA[22]; (d) The laser operating above threshold, a nonlinear many-atom case.[23]

We will study (a) and (c) in detail, comment briefly on (b), and allude to well-known results for (d). The statistical effects vary among these models and consideration of them will help us to understand the consequences of stimulated emission (when it can be identified) and of wave interference for the evolution of photon statistics.

Finally, Glauber[24] has shown that the quantum superposition of many independent fields leads to BE statistics. In those cases in which the stimulated emission leads to coherent (linear) amplification and therefore does not change the photocount statistics (i.e., SA and UL), the spontaneous emissions act as this source of many independent waves. In those cases involving nonlinear amplification, the interaction does not factor and thus the change in the statistics cannot be attributed to the separate action of stimulated or spontaneous emission. It is easily shown, however, that BE statistics are either not present or are not conserved in time by these processes.

We therefore conclude that wave interference effects give rise to the BE statistics of light. In those cases characterized by nonlinear interactions, BE statistics typically do not occur.

II. PHYSICAL AND MATHEMATICAL BACKGROUND

For simplicity, we consider a single mode of the electromagnetic field, for which a and $a^\dagger$ are the annihilation and creation operators. The atoms interacting with the field possess a significant electric dipole matrix element for transitions between two energy levels whose energy separation $\hbar\omega$ is equal to the energy of one photon of the field. We consider them to be (for our purposes) two-level atoms for which the raising and lowering operators are σ^+ and σ^-, respectively. The interaction Hamiltonian for each atom in the rotating-wave approximation (RWA) is assumed to be

$$H_I = \hbar g (a\sigma^+ + a^\dagger\sigma^-) \ , \qquad (2)$$

in which the coupling constant g contains both the electric-dipole matrix element and appropriate dependence on the atomic positions in the mode of the field which is being considered.[1,10,11] Hence, we are considering the one-photon interaction with a resonant medium consisting of one or more atoms, whose state will be specified more fully below.

To fix notation, we now define the appropriate statistical quantities with which we will deal. The density matrix of the electromagnetic field ρ contains complete information about the time evolution of the field statistics. This is defined to be the reduced density matrix obtained from the complete atom-field density matrix by tracing over the atomic variables. The quantum characteristic function $C(\zeta, \eta)$ was defined above. It contains all the information which is contained in the reduced density matrix. Arbitrary moments of the creation and annihilation operators, such as

$$\langle (a^\dagger)^m a^n \rangle \equiv \mathrm{tr}[\rho (a^\dagger)^m a^n] ,$$

may be obtained by mth and nth order differentiations with respect to the arguments of $C(\zeta, \eta)$ and then setting the arguments equal to zero. As will be shown below, in certain cases the characteristic function is the most convenient for determining the evolution of the photon statistics since interference effects arising from the superposition of independent fields can be disentangled. Familiarity with the characteristic functions of various distributions allows one to identify the statistics of the field.

In certain cases it is more convenient to derive the photon-number generating function. Although it contains information only about the diagonal elements of the density matrix, it is just this information which is sometimes most readily interpreted. The generating function is defined by

$$G(z, t) = \mathrm{tr}[\rho z^{a^\dagger a}] \equiv \langle z^{a^\dagger a} \rangle = \sum_{n=0}^{\infty} \rho_n z^n , \tag{3}$$

in which convergence is guaranteed for $|z| \leq 1$. The factorial moments, which are related to the moments of the intensity[25] (when they can be defined), are obtained by repeated differentiation with respect to z, followed by setting $z = 1$. The availability of the factorial moments allows one to obtain information concerning the field-intensity fluctuations. In the case of superposition of independent fields, however, the photon-number generating function does not factor because of interference effects. Thus, it is not always possible to identify separate contributions from examination of the generating function.[26]

In the following, we shall investigate various models of stimulated and spontaneous emission and obtain the differential equations satisfied by the quantum characteristic function, and other differential equations and relations satisfied by the generating functions. Solving these differential equations, we will have the desired information concerning the field statistics.

III. UNSATURATED LASER AMPLIFIER

We first consider the many-atom case in which the populations of the upper and lower atomic states are held constant by a pumping process. This is a model of a laser amplifier operating far below threshold so that saturation effects are not important. The time evolution of the radiation field density matrix is determined by[22]

$$\dot{\rho}(t) = \tfrac{1}{2}\{A(a^\dagger \rho a - a a^\dagger \rho) + B(a \rho a^\dagger - a^\dagger a \rho)\} + (\text{adjoint}) , \tag{4}$$

where $A = 2\Gamma^{-1} g^2 N_a$, $B = 2\Gamma^{-1} g^2 N_b$, and Γ is the width of the transition between the upper and lower states, whose populations are N_a and N_b, respectively.

The evolution of the diagonal matrix elements $\rho_n = \langle n|\rho|n\rangle$ obtained from Eq. (4), is given by

$$\dot{\rho}_n = -A(n+1)\rho_n - Bn\rho_n + An\rho_{n-1} + B(n+1)\rho_{n+1} . \tag{5}$$

It has been shown by several authors that this equation preserves the statistics of a Bose-Einstein distribution.[7,11,17,18] In seeking the source of the BE correlations, various terms in Eq. (5) have been associated with either spontaneous or stimulated emission. One suggestion for the evolution of the diagonal terms in the density matrix resulting from the stimulated emission only,[2]

$$\dot{\rho}_n = -An\rho_n - Bn\rho_n + A(n-1)\rho_{n-1} + B(n+1)\rho_{n+1} , \tag{6}$$

is obtained by replacing n by $n-1$ in the coefficients of A in Eq. (5).

Because the total transition rate from n to $n+1$ photons is well known to be proportional to $n+1$, where the 1 is associated with spontaneous emission, the above replacement supposedly deletes the effects of spontaneous emission. The terms in Eq. (5) which are missing from Eq. (6) presumably give the rate of increase of the spontaneous-emission field. This yields, for the spontaneous emission alone,

$$\dot{\rho}_n = -A\rho_n + A\rho_{n-1} . \tag{7}$$

This is recognized as the Kolmogorov equation[27] for the Poisson process which describes the independent emission of noninteracting classical particles. It is seen that in the attempt to remove stimulated emission the interference effects expected from independently emitted fields have also been lost. Further analysis in Appendix A shows that the generating function for this process may be written

$$G(z, t) = e^{(z-1)\bar{n}} G_0(z) , \tag{8a}$$

which for an initial vacuum $G_0 = 1$ has the form

$$G(z, t) = e^{(z-1)\bar{n}} , \tag{8b}$$

which is the generating function for a Poisson distribution of mean $\bar{n}=At$. The result in Eq. (8a) is then the product of the generating function for the initial field and the generating function for the Poisson distribution expected when there is no initial field. The factoring of Eq. (8a) implies that the spontaneous emission defined by Eq. (7) does not interfere with the initial field. Thus, ultimately, this identification of terms cannot be accepted because spontaneous emission so defined does not behave as a field.

In this model, it has also been maintained that stimulated emission, according to Eq. (6), changes the statistics of (and adds excess correlations to) any electromagnetic field, whether originating from spontaneous emission alone or including the contribution of an incident field.[1,2] That Eq. (6) does indeed change the statistics of an arbitrary field is shown in Appendix A by solving for the generating function for this case,

$$G(z,t)=G_0\left(\frac{1-B[(z-1)/(Az-B)]e^{\gamma t}}{1-A[(z-1)/(Az-B)]\,e^{\gamma t}}\right), \tag{9}$$

where $\gamma=A-B$ and $G_0(z)$ is the generating function for the initial-field statistics. Clearly, the statistics of the initial field are, in general, changed by the interaction. In fact, the relative fluctuations (defined as the normalized second factorial moment) always increase for $A\neq 0$. (See Appendix A.) Specifically, one sees that a field which is initially thermal does not remain so. A geometric (BE) distribution of photons has a generating function of the form

$$G(z)=1/[1-(z-1)\bar{n}], \tag{10}$$

in which $\bar{n}$ is the average number of photons. Taking this form for G_0 in Eq. (9), one readily verifies that $G(z,t)$ does not remain of the form given by Eq. (10). Similarly, one can easily show by using Eq. (8b) in Eq. (9) that a field which is initially Poisson, as is the case for the "spontaneous emission" defined by Eq. (7), does not become geometric as it evolves according to Eq. (9).[28]

The major shortcoming of the identification of stimulated and spontaneous emission as given by Eqs. (6) and (7) is that while the Poisson distribution of spontaneous photons may seem reasonable from the standpoint of isolated atoms independently emitting photons, it is not consistent with the wave character of the electromagnetic field, which is well known to give rise to excess photocount correlations as a result of the superposition of independent contributions to the field.

Before proceeding to the full solution of the density-matrix equation, we will consider an alternative identification of terms.[7,17] Within the Fokker-Planck equation satisfied by $P(\alpha)$, rather than in the number representation, terms giving rise to coherent amplification have been identified with stimulated emission and other terms generating Gaussian noise with spontaneous emission. Transforming back into the number representation leads to the following regrouping of terms in Eq. (5)[7]

$$\dot{\rho}_n=(A-B)[n\rho_n-(n+1)\rho_{n+1}] +A[(n+1)\rho_{n+1}-(2n+1)\rho_n+n\rho_{n-1}]. \tag{11a}$$

Here the terms proportional to $A-B$ and to A describe amplification and noise generation, respectively. Because of the independence of A and B, this regrouping uniquely determines a regrouping of the operator equation [Eq. (4)] as follows[29]

$$\dot{\rho}=\tfrac{1}{2}(A-B)[(a^\dagger a\rho-a\rho a^\dagger)+(\text{adjoint})] +\tfrac{1}{2}A[(a^\dagger\rho a-(2a^\dagger a+1)\rho+a\rho a^\dagger)+(\text{adjoint})]. \tag{11b}$$

The effects of the terms in Eq. (11a) have been explored, both separately and jointly, for fields having valid P representations.[7] In this separation, the term proportional to $A-B$ causes linear amplification (or attenuation) and is associated with stimulated emission (or absorption). The term proportional to A is associated with spontaneous emission. (Note that in this separation if $A=B$ there are no effects of stimulated emission or absorption.)

We wish to consider whether these meanings remain valid for any initial field. In Appendix C, we derive the generating functions for the field statistics as they evolve according to the two terms in Eq. (11a). For the spontaneous emission part of Eq. (11a) we obtain

$$G(z,t)=\frac{1}{1-(z-1)At}G_0\left(\frac{(z-1)+1-(z-1)At}{1-(z-1)At}\right), \tag{12a}$$

which for an initial vacuum has the form

$$G(z,t)=1/[1-(z-1)At]. \tag{12b}$$

This is indeed the generating function for all times for a geometric (BE) distribution of photons. Note that $\bar{n}=At$ since there is no amplification of the spontaneous emission. The complicated argument in the G_0 factor for $t>0$ in Eq. (12a) indicates that spontaneous emissions so defined are not describable merely as added independent classical particles as was the result of Eq. (7).

The stimulated emission part of Eq. (11a) has the form

$$\dot{\rho}_n=(A-B)[n\rho_n-(n+1)\rho_{n+1}]. \tag{13}$$

The generating function for this simulated emission process is obtained in Appendix C with the result

$$G(z,t)=G_0[(z-1)e^{\gamma t}+1]. \tag{14}$$

This implies for the mth factorial moment that

$$\left(\frac{d}{dz}\right)^m G(z,t)\Big|_{z=1} = e^{m\gamma t}\left(\frac{d}{dz}\right)^m G_0(z)\Big|_{z=1}. \quad (15)$$

Hence, the normalized factorial moments, e.g.,

$$\frac{\langle n(n-1)(n-2)\cdots(n-m+1)\rangle}{\langle n\rangle^m},$$

do not change in time. The nature of the photon statistics of an arbitrary initial field are thus unchanged as a result of this stimulated emission. For example, this can be seen by using Eq. (10) or Eq. (8b) for G_0 in Eq. (14). This result corresponds exactly to the constancy in time of the normalized moments of α in the $P(\alpha)$ representation under the action of this term.[7,14]

Equation (13) may lead to probabilities which are negative or greater than one in certain cases. If $A > B$ and if $\rho_{n+1} \neq 0$, while $\rho_n = 0$, then clearly ρ_n goes negative. This may be seen to result because Eq. (13) is not a valid Kolmogorov equation[27,30] when $A > B$ because $\lambda_{nn} > 0$. Thus it is nonsense, in general, to consider having this stimulated emission exist alone. The precise limitation has been derived in recent work[31–33] which has shown that a field can be arbitrarily amplified, preserving its normalized statistics, if and only if $P(\alpha)$ is non-negative. In real situations there is no problem, as indicated in Eq. (11a) where stimulated and spontaneous effects necessarily appear together forming a valid equation. Consistent with results from Refs. 31–33, we also note that if $P(\alpha)$ exists and is non-negative, then the relation

$$\rho_n = \int |\alpha|^{2n}(n!)^{-1}e^{-|\alpha|^2}P(\alpha)d^2\alpha \quad (16)$$

implies ρ_n is also non-negative. Thus, these stimulated-emission terms may formally be considered to act alone in these cases without leading to unphysical results. These are also just those cases for which the field has a classical analog.

We have seen that the separate action of the stimulated-emission term in this model may not be valid for fields which do not possess a non-negative P representation. That the derivation of this model relied on the existence of the P representation is a possible objection to this identification of spontaneous and stimulated emission.

To deal with this limitation, we have derived in Appendix D the generating function for the complete Eq. (5) which governs the evolution of the full field density matrix

$$G(z,t) = \frac{1}{1-(z-1)\bar{n}_s} \times G_0\left[\frac{1-B[(z-1)/(Az-B)]e^{\gamma t}}{1-A[(z-1)/(Az-B)]e^{\gamma t}}\right], \quad (17)$$

where

$$\bar{n}_s = A\gamma^{-1}(e^{\gamma t}-1). \quad (18)$$

From Eq. (17) we see the well-known result that if the initial field is the vacuum ($G_0 = 1$) then the field is BE for all time since then $G(z,t)$ is the generating function for a geometric distribution for all times, with an average number of photons given by Eq. (18).

Equation (17) completely specifies the evolution of the photon statistics for an arbitrary initial field. Although the various factorial moments may be obtained from Eq. (17) by differentiation with respect to z, an easier method for calculating the factorial moments is described in Appendix D. There we make use of the differential equation satisfied by $G(z,t)$ to obtain the differential equations satisfied by the factorial moments. In this way we obtain, for instance, the average number of photons as a function of time,

$$\bar{n}(t) = \bar{n}_s + \bar{n}_0 e^{\gamma t}, \quad (19)$$

where $\bar{n}_0$ is the average number of photons in the initial field described by G_0. Thus, the average number of photons is equal to those in the amplified spontaneous emission ($\bar{n}_s$) plus those in the amplified input field ($\bar{n}_0 e^{\gamma t}$). In a similar manner, one can easily solve for the second and higher factorial moments as a function of time.[34]

Although we have analyzed the action of certain stimulated and spontaneous-emission source terms separately, when both terms act simultaneously we cannot immediately interpret the result given by Eq. (17).[26,35]

In order to explore the possibility that the final field is the superposition of several fields, we must turn from the generating function to the quantum characteristic function which is derived in Appendix E. The characteristic function as defined will factor in the case of superposition of independent fields.

We obtain for the full interaction described by Eq. (4)

$$C(\zeta,\eta) = e^{\bar{n}_s\zeta\eta}C_0(\zeta e^{\gamma t/2}, \eta e^{\gamma t/2}), \quad (20)$$

in which $\bar{n}_s$ is as defined in Eq. (18) and $C_0(\zeta,\eta)$ is the quantum characteristic function of the initial field.

The first factor alone results if the initial field is the vacuum ($C_0 = 1$) and therefore must represent amplified spontaneous emission (ASE). Choosing $\zeta = i\theta$ and $\eta = i\theta^*$ to make closer contact with the ordinary characteristic function, we obtain

$$C(\theta) = \exp{-\bar{n}_s|\theta|^2}. \quad (21)$$

This is the standard form for the characteristic function of a Gaussian random variable with mean zero and variance equal to $2\bar{n}_s$. This is consistent with the results obtained from Eq. (17) when $G_0=1$, with prior work in the P representation,[7] and with other earlier work.[9,11]

The second factor in Eq. (20) describes the coherent amplification or attenuation of the input field given by C_0. In direct correspondence to our Eq. (15) and many similar results,[7,9,11,13] we see that the normalized moments of the amplified input field are unchanged since

$$\left(\frac{d}{d\zeta}\right)^j\left(\frac{d}{d\eta}\right)^k C_0(\zeta e^{\gamma t/2},\eta e^{\gamma t/2})\bigg|_{\zeta=\eta=0} = e^{(j+k)\gamma t/2}\langle (a^\dagger)^j a^k\rangle_0\,, \quad (22)$$

where the subscript zero in Eq. (22) means that the expectation value is taken with respect to the density matrix of the input field.

Thus, clearly, the final field is the superposition of two fields which may be identified as amplified spontaneous emission and a field resulting from coherent stimulated amplification (or attenuation) of the input field.

We show in Appendix E that the two factors in Eq. (20) have as their fundamental sources the stimulated and spontaneous emission terms identified in Eq. (11b). The first factor describes the spontaneously emitted field which has been subsequently amplified. The second factor in Eq. (20) is derived directly from the stimulated emission terms acting alone.

For the action of the spontaneous-emission terms in Eq. (11b) we have the result

$$C(\zeta,\eta)=e^{At\zeta\eta}C_0(\zeta,\eta)\,. \quad (23)$$

Note that this characteristic function corresponds to the generating function displayed in Eq. (12a) since they describe the same field. Equation (23) shows that this field is just the superposition of the unchanged initial field and a spontaneously emitted field identical with that generated in an initial vacuum and described by

$$C(\zeta,\eta)=e^{At\zeta\eta}\,. \quad (23a)$$

The spontaneously emitted field is also Gaussian, but the mean reflects linear growth in time in contrast to the time dependence of the mean of the ASE described by Eq. (18).

We have not only shown that the effects represented by the two factors in Eq. (20) are characteristic of spontaneous and stimulated emission, but we have proven that these effects have as their source the terms in Eq. (11b). Thus, we may conclude that the terms in Eq. (11b) represent the action of stimulated and spontaneous emission. These results confirm Abraham and Smith's identification of terms which was based on the Fokker-Planck equation for $P(\alpha)$, but our results are more general in that they apply to arbitrary initial fields.

The form of Eq. (20), the product of two independent terms describing the evolution of the ASE and the amplified input field, indicates that we may view these processes as simultaneous independent stochastic processes. This was concluded by Abraham and Smith under less general conditions in the P representation. Although the stimulated-emission terms cannot always stand alone, they are in fact always accompanied by the spontaneous-emission terms, and the full equation always leads to acceptable results.

We have thus shown that the evolution of an arbitrary input field results from coherent amplification and the addition of Gaussian noise. In particular, the amplification of initial spontaneous emission must be coherent. Therefore, since we have shown that the ASE is BE, it must have originated as a BE distribution. This is the theoretical result promised earlier—the BE nature of the ASE (and thus the correlations) have their origin in the BE nature of the original spontaneous emission.

Further understanding of the spontaneous emission terms in Eq. (11a) can come from the following extension of Glauber's[24] derivation of BE statistics and the Gaussian distribution of the field amplitude from the superposition of contributions from a large number of independent sources. Assuming only that each contribution could be represented by the same $p(\alpha)$ or that all of the P representations had comparable moments, Glauber found that the superposition of a large number of contributions could be written

$$P(\alpha)=\frac{1}{\pi N\langle|\alpha|^2\rangle}\exp\left(\frac{-|\alpha|^2}{N\langle|\alpha|^2\rangle}\right).$$

If, rather than considering a static situation, we take a system with a constant rate of contributions so that

$$N=\beta t\,,$$

then $P(\alpha)$ satisfies the following differential equation

$$\frac{\partial}{\partial t}P(\alpha)=\beta\langle|\alpha|^2\rangle\frac{\partial^2}{\partial\alpha\partial\alpha^*}P(\alpha)\,.$$

This is just the Fokker-Planck equation for diffusion in two dimensions.[36]

An even simpler and more intuitive feeling for the source of the diffusion term results if each $p(\alpha)\geq 0$. Then $p(\alpha)$ is analogous to a probability density for α and the evolution of $P(\alpha)$ is analogous to a classical two-dimensional random walk in the α-plane. In the limit of many small steps, such a random walk is equivalent to diffusion in two dimensions,[37] as described by the well-known diffusion term in the Fokker-Planck equation. It is this diffusion equation for $P(\alpha)$, or the corresponding equation for $\hat{\rho}_n$ which results when the diffusion equation is transformed by Eq. (16), which has been identified with spontaneous emission.

The mathematical steps of this derivation are almost identical with those of Abraham and Smith, but the interpretation is more immediate. We see that the spontaneous-emission terms can be understood as resulting from the addition of many independent interfering contributions. The origin of BE correlations, therefore, is just the random superpositions and interference of fields resulting from any chaotic source, spontaneous emission from a large number of uncorrelated atoms being one example.

IV. SINGLE ISOLATED ATOM

We now consider the case of a single isolated atom interacting with a resonant mode of the field, as described in Sec. II. Our purpose, as discussed in Sec. I, is to compare this case with the others listed in Sec. I, in particular with that considered in Sec. III, so as to emphasize the differences between the two physical models. By carrying out exact calculations for the evolution of the photon statistics, we will show that (i) the evolution of the statistics is, not surprisingly, quite different from that of the laser amplifier, and (ii) even BE statistics are not maintained after the one-atom interaction.

We now calculate the generating function for this case. Following Ref. 1 we will assume that the atom is initially in the excited state. The initial state of the field is described by the density matrix $\rho(0)$. Although our derivation applies whether or not the initial field is diagonal in the Fock representation, we will only be concerned with the diagonal elements of the density matrix.

Given that there are n photons present in the field at time t, there are only two possibilities. Either there were $n-1$ photons at time zero and the atom is now in the ground state, having emitted the additional photon, or there were n photons at time zero and the atom is still in the excited state. That is,

$$\rho_n(t)=\rho_{n-1}(0)|C_{b,n}(t)|^2+\rho_n(0)|C_{a,n}(t)|^2\,, \qquad (24)$$

where $C_{b,n}(t)$ is the (conditional) probability amplitude for the atom to be in the ground state and the field to be in the n-photon state (having started with $n-1$ photons) with the initial condition $C_{b,n}(0)=0$. Similarly, $C_{a,n}(t)$ is the probability amplitude for the atom to be in the upper level and the field to be in the n-photon state (given initially there were n photons), with the initial condition $C_{a,n}(0)=1$.

The required probability amplitudes (RWA) can be solved for exactly within the present model and correspond to Rabi's flopping atom in a quantized field.[38] Using the well-known probabilities that the atom is in the upper or lower state, we obtain for the field-density matrix,

$$\rho_n(t)=\rho_{n-1}(0)\sin^2(g\sqrt{n}\,t)+\rho_n(0)\cos^2[g(n+1)^{1/2}t]\,. \qquad (25)$$

Now, using $\cos^2+\sin^2=1$ and the definition of the generating function, Eq. (3), we obtain

$$G(z,t)=G_0(z)+(z-1)\langle\sin^2[g(n+1)^{1/2}t]z^n\rangle_0\,, \qquad (26)$$

where again the zero subscript refers to the initial distribution.

By taking derivatives with respect to z we obtain the first and second factorial moments,

$$\langle n\rangle_t=\langle n\rangle_0+\langle\sin^2[g(n+1)^{1/2}t]\rangle_0\,, \qquad (27)$$

and

$$\langle n(n-1)\rangle_t=\langle n(n-1)\rangle_0+2\langle n\sin^2[g(n+1)^{1/2}t]\rangle_0\,. \qquad (28)$$

From Eq. (27) we see that the rate of emission by the atom is a function of the field statistics.

While Eq. (26) clearly indicates that the factorial moments will be the sum of the initial value and a time-dependent contribution due to the interaction, it is difficult to identify separate effects of stimulated and spontaneous emission in these equations. The effects of stimulated and spontaneous emission might be associated by some with the terms n and 1, respectively, in the factor $(n+1)^{1/2}$ (or, similarly, the $n-1$ and 1 in $\sqrt{n}$) which appears in the upper and lower state amplitudes of the flopping atom. Considered to lowest order in time, however, the square root disappears and Eq. (25) becomes

$$\rho_n(t)=\rho_{n-1}(0)(gt)^2n+\rho_n(0)-\rho_n(0)(gt)^2(n+1)\,, \qquad (29)$$

which is just Mandel's Eq. (20).[1] We note that it is of the same functional form as Eq. (5), though with $B=0$ and a quadratic time dependence. We see that the n and 1, identified in Ref. 1 with stimulated and spontaneous emission, are derived from the $(n+1)^{1/2}$ factor. Such an identification will lead to the unphysical results investigated in Sec. III for the corresponding separation of terms.

From Eq. (26) we see that the nature of the generating function changes in general and hence that the statistics are not conserved. In particular we show below that BE statistics will be altered by the full atom-field interaction, although it is correct that to lowest order in time there is no change in the normalized excess correlation for initially BE statistics, as shown in Ref. 1. From Eq. (26) or from what follows below one could prove the stronger statement that to lowest order in time the interaction preserves BE statistics. This also follows from the similarity between Eqs. (29) and (5) and the properties of the latter equation as already discussed.

If the initial field has BE statistics, then $\rho_n(0)$ is geometric, i.e.,

$$\rho_n(0) = (1-\beta)\beta^n \,, \tag{30}$$

where

$$0<\beta<1 \,.$$

It is necessary that $\rho_n(t)/\rho_{n-1}(t)$ be a constant if the field is to have BE statistics at time t. Using Eqs. (25) and (30) we obtain

$$\frac{\rho_n(t)}{\rho_{n-1}(t)} = \frac{\beta^2\cos^2[g(n+1)^{1/2}t] + \beta\sin^2(g\sqrt{n}\,t)}{\beta\cos^2(g\sqrt{n}\,t) + \sin^2[g(n-1)^{1/2}t]} \,. \tag{31}$$

The right-hand side of Eq. (31) is obviously not a constant. Since BE statistics are not maintained under this interaction, it is difficult to justify the position that the interaction leads to BE statistics.

Hence, either by observing the form of the exact generating function, Eq. (26), or by using Eqs. (27) and (28) to calculate the excess correlations to any desired order in t, we verify that there are indeed changes in the photon statistics arising from the interaction. However, there are no separately identifiable spontaneous or stimulated emission effects.

It is important to note that this single atom is not being maintained in the upper state. The changing of the populations of atomic levels in response to the ambient field naturally gives rise to nonlinear effects.

In conclusion, we have found that, in general, the single-atom interaction does indeed change the correlations of an initial field. However, it does not conserve BE statistics and thus cannot be considered the source of BE correlations.

V. DISCUSSION

An excited atom radiating into an incident field will be influenced by the presence of the ambient field. The distribution of photons in the evolving field will reflect not only the enhanced rate of emission, identified with stimulated emission, but also the effects of interference with spontaneous emission expected from the wave character of the electromagnetic field. One might expect the latter even if the atom were not influenced by the field. One could perhaps argue that it is just this interference aspect which is responsible for drawing energy out of the atom at the stimulated rate, and hence that these two effects are indistinguishable. According to this latter point of view, any changes in evolution of the field statistics resulting from the addition of an incident field would presumably be explained as resulting from stimulated emission. It would appear that several previous authors[1-3] subscribe, at least implicitly, to this type of reasoning since they do not separate the effects of interference from those of stimulated emission.

Abraham and Smith have explicitly separated these effects. Our alternate derivation, in Sec. III, of their spontaneous-emission rate equation graphically confirms the correctness of separating interference and stimulated effects. We are thus led to the conclusion that by deleting what are customarily identified as terms arising from stimulated emission, one loses not only stimulated emission, but also all interference effects. This explains, for instance, why Webb[3] incorrectly concluded that "stimulated emission is responsible for photons obeying BE statistics in blackbody radiation," as was pointed out by Abraham and Smith. This also explains why, as we showed in Sec. III, Vorobev and Sokolovskii's attempt to separate out stimulated effects leads to classical particle (noninterfering, Poisson distribution) effects for the spontaneous emission [cf. Eq. (8)].

Interference effects and the properties of random superposition of independent increments of the field amplitude (as characterized by the well-known results of the central-limit theorem, random walk, and diffusion in a plane) are sufficient to explain the results of the Hanbury Brown–Twiss experiment and the presence of BE statistics in those cases in which we have shown that the amplification is linear and, therefore, coherent. In those other cases in which nonlinearities are important, such as for the single isolated atom or in a laser, the changes in the photon statistics are such that BE statistics are not conserved (as we showed in Sec. IV for the former and as is well known for the latter case).

ACKNOWLEDGMENT

This work was supported in part by the National Science Foundation under Grant No. ENG 77-01118.

APPENDIX A

In this appendix, we will derive the generating function, defined by Eq. (3), for the photon-number probabilities (the diagonal elements of the density matrix) which evolve according to Eqs. (6) and (7). We obtain the partial differential equations satisfied by $G(z,t)$ by multiplying each equation by z^n and summing over n from zero to infinity. A term of the form

$$\sum_{n=0}^{\infty} n\rho_n z^n$$

may be seen to be equal to

$$z\frac{d}{dz}G(z,t)=z\frac{d}{dz}\sum_{n=0}^{\infty}\rho_n z^n .$$

Similarly, a term arises of the form

$$\sum_{n=0}^{\infty}(n-1)\rho_{n-1}z^n .$$

Since the diagonal elements of the density matrix are defined to be zero for n less than zero, the above summation equals

$$z^2(d/dz)G .$$

Hence, we obtain for Eq. (6)

$$\frac{d}{dt}G(z,t)=-Az\frac{d}{dz}G-Bz\frac{d}{dz}G+Az^2\frac{d}{dz}G+B\frac{d}{dz}G ,$$

which is equal to

$$\frac{d}{dt}G(z,t)=(Az-B)(z-1)\frac{d}{dz}G(z,t). \tag{A1}$$

We solve this partial differential equation by observing that if we can find a function y of z such that

$$dz/dy=(Az-B)(z-1), \tag{A2}$$

then Eq. (A1) reduces to

$$\left(\frac{d}{dt}-\frac{d}{dy}\right)G=0. \tag{A3}$$

The solution of Eq. (A3) is an arbitrary function of $y+t$. Expanding dy/dz in partial fractions and integrating, we obtain (the constant of integration is set to zero for convenience)

$$y=-\frac{1}{\gamma}\ln(Az-B)+\frac{1}{\gamma}\ln(z-1)$$

$$=\frac{1}{\gamma}\ln\left(\frac{z-1}{Az-B}\right),$$

where $\gamma=A-B$.

Therefore, G is an arbitrary function of $y+t$ or, what is the same thing, of

$$x=e^{\gamma(y+t)}=\left(\frac{z-1}{Az-B}\right)e^{\gamma t}. \tag{A4}$$

The form of the arbitrary function is determined by the initial condition

$$G(z,0)=G_0(z). \tag{A5}$$

Consequently, we must find a function of the quantity in Eq. (A4) which reduces to z when t is zero. This is found by setting $t=0$ and solving Eq. (A4) for z in terms of x. After a little algebra, we find the required argument of G_0, and hence

$$G(z,t)=G_0\left(\frac{1-B[(z-1)/(Az-B)]e^{\gamma t}}{1-A[(z-1)/(Az-B)]e^{\gamma t}}\right). \tag{A6}$$

If desired, one can now obtain the quantities

$$\rho_n(t)=\frac{1}{n!}\left(\frac{d}{dz}\right)^n G(z,t)\bigg|_{z=0}. \tag{A7}$$

We also note that the kth factorial moment

$$\langle n(n-1)\cdots(n-k+1)\rangle=\left(\frac{d}{dz}\right)^k G(z,t)\bigg|_{z=1} \tag{A8}$$

can be found by the method of differential equations described in Appendix D, with the result for the first two factorial moments that

$$d\bar{n}/dt=\gamma\bar{n}, \tag{A9}$$

and

$$\frac{d\langle n(n-1)\rangle}{dt}=2A\bar{n}+2\gamma\langle n(n-1)\rangle. \tag{A10}$$

Using these results, we see for the normalized excess fluctuations ξ that

$$\frac{d\xi}{dt}\equiv\frac{d}{dt}\left(\frac{\langle n(n-1)\rangle-\bar{n}^2}{\bar{n}^2}\right)=\frac{d}{dt}\left(\frac{\langle n(n-1)\rangle}{\bar{n}^2}\right)=\frac{2A}{\bar{n}}. \tag{A11}$$

Thus stimulated emission as identified in Eq. (6) increases the correlations unless $A=0$.

The spontaneous-emission terms from Eq. (7) lead to the following equation for the generating function.

$$\frac{d}{dt}G(z,t)=-AG(z,t)+zAG(z,t)$$

$$=A(z-1)G(z,t). \tag{A12}$$

The solution for spontaneous emission in this model is

$$G(z,t)=e^{At(z-1)}G_0(z). \tag{A13}$$

APPENDIX B

In this Appendix, we will explore how far one can pursue the model suggested by Eqs. (6) and (7). The contributions of the amplified spontaneous emissions originating at different times will be added, based on Eq. (8a), as if the numbers of photons were classical independent random variables (which is not the case). The model for amplification used here is that suggested by Vorobev and Sokolovskii,[2] given in our Eq. (6), and discussed in Appendix A. Our purpose is to see in what manner one might understand their assertion (and Mandel's similar position[1]) that the change in statistics brought about by Eq. (6) gives rise to BE photon statistics. We assume for argument's sake that the spontaneous emission gives rise [in accordance with Eq. (8b)] to a Poisson distribution in each small time interval. For simplicity, we assume that $B=0$, i.e., that there is no absorption.

In that case, Eq. (9) for the "amplification" process reduces to

$$G(z,t)=G_0\left(\frac{1}{1-[(z-1)/z]e^{At}}\right). \tag{B1}$$

The generating function of the sum of independent random variables is given by the product of their generating functions.[39] Let $G_{\Delta t}(z,k\Delta t)$ be the generating function for those photons spontaneously emitted during a time interval Δt at time $k\Delta t$ ago, and all photons arising from them. The integer parameter k varies between 1 and N, where $t=N\Delta t$. Time could also be interpreted as distance along an amplifier tube. Each $G_{\Delta t}$ is of the form of equation (B1).

The generating function for all photons is then

$$\bar{G}(z,t)=\prod_{k=1}^{N}G_{\Delta t}(z,k\Delta t). \tag{B2}$$

According to Eq. (8b), the distribution of spontaneously emitted photons in Δt is Poisson, with mean $A\Delta t$. Hence we have

$$G_0(z)=e^{A\Delta t(z-1)}. \tag{B3}$$

Now, using the above expressions in Eq. (B2), we find

$$\begin{aligned}\lim_{\substack{N\to\infty\\ \Delta t\to 0}}\bar{G}(z,t)&=\lim_{\substack{N\to\infty\\ \Delta t\to 0}}\prod_{k=1}^{N}\exp\left[A\Delta t\left(\frac{1}{(1-[(z-1)/z]e^{Ak\Delta t})}-1\right)\right]\\ &=\lim_{\substack{N\to\infty\\ \Delta t\to 0}}\exp\left(-A\sum_{k=1}^{N}\Delta t\right)\\ &\quad\times\exp\left[A\sum_{k=1}^{N}\left(\frac{\Delta t}{1-[(z-1)/z]e^{Ak\Delta t}}\right)\right],\end{aligned} \tag{B4}$$

where the limits are taken such that $N\Delta t=t$. Using the definition of the Riemann integral, Eq. (B4) becomes

$$\bar{G}(z,t)=\exp\left(A\int_0^t\frac{dt'}{1-[(z-1)/z]e^{At'}}\right)\times\exp\left(-A\int_0^t dt'\right). \tag{B5}$$

The integrals are standard, and we obtain

$$\bar{G}(z,t)=\frac{1}{1-(z-1)(e^{At}-1)}, \tag{B6}$$

which is indeed the generating function for a geometric BE distribution of photons with mean $e^{At}-1$. Of course, this result does not justify the model implied by Eqs. (6) and (7) because the model is based on noninterfering emissions.

APPENDIX C

We now derive the generating functions that result from the stimulated and spontaneous emission terms suggested by Abraham and Smith. [See our Eq. (11a).] For spontaneous emission alone (or when $A=B$), the density matrix evolves according to

$$\frac{d}{dt}\rho_n=A[(n+1)\rho_{n+1}-(2n+1)\rho_n+n\rho_{n-1}]. \tag{C1}$$

We first multiply Eq. (C1) by z^n and sum over n from zero to infinity. Using techniques similar to those used in Appendix A for terms of the form $n\rho_n z^n$ and $(n+1)\rho_{n+1}z^n$, etc., we obtain the partial differential equation governing the evolution of $G(z,t)$,

$$\frac{d}{dt}G(z,t)=A(z-1)\left((z-1)\frac{d}{dz}+1\right)G(z,t). \tag{C2}$$

Equation (C2) is quite similar in form to the equation for G which describes the evolution of the density matrix according to the full Eq. (5). We refer to Appendix D, where that equation for G is solved more generally, for the details of the solution of Eq. (C2). Alternatively, one can verify by substitution that Eq. (C2) is satisfied by

$$G(z,t)=\frac{1}{1-(z-1)At}G_0\left(\frac{(z-1)+1-(z-1)At}{1-(z-1)At}\right), \tag{C3}$$

which reduces to

$$G(z,t)=1/[1-(z-1)At] \tag{C4}$$

for an initial vacuum state.

Turning now to the terms suggested by Abraham and Smith as giving rise to amplification through stimulated emission, we consider Eq. (13)

$$\dot{\rho}_n=(A-B)[n\rho_n-(n+1)\rho_{n+1}].$$

We again multiply by z^n and sum over n. Using methods identical to those in Appendix A, we obtain

$$\frac{d}{dt}G(z,t)=\gamma(z-1)\frac{d}{dz}G(z,t), \tag{C5}$$

which has the solution

$$G(z,t)=G_0[(z-1)e^{\gamma t}+1]. \tag{C6}$$

This result is discussed following Eq. (14). Note that if $\gamma>0$, the argument of G_0 does not, in general, have absolute value less than one, as was required of z to ensure convergence of the sum defining G_0. In general, this will create problems reflected in the behavior of $G(z,t)$. [See the discussion before and after Eq. (16).] Why no problem arises if there exists a non-negative P representation can be understood by multiplying Eq. (16) by z^n and summing. We obtain

$$G_0(z)=\int e^{|\alpha|^2(z-1)}P(\alpha)d^2\alpha. \tag{C7}$$

Using this in Eq. (C6) results in

$$G(z,t)=\int e^{|\alpha|^2 e^{\gamma t}(z-1)}P(\alpha)d^2\alpha\,, \tag{C8}$$

and a change of scale in the α plane brings us back to the original form of Eq. (C7) (with a different but still non-negative P representation) which by assumption is convergent. (Note: $|z|\leqslant 1$, $P(\alpha)\geqslant 0$, and $\int P(\alpha)d^2\alpha=1$.)

APPENDIX D

In this Appendix, we will derive the generating function corresponding to the complete density matrix which evolves in accordance with Eq. (5). Making use of methods described in Appendixes A and B, we multiply Eq. (5) by z^n and sum over n, obtaining

$$\frac{d}{dt}G(z,t)=(z-1)\left((Az-B)\frac{d}{dz}G+AG\right)$$
$$=(z-1)(Az-B)\left(\frac{d}{dz}G+\frac{A}{(Az-B)}G\right). \tag{D1}$$

The expression in the large parentheses is equal to

$$e^{-\ln(Az-B)}\frac{d}{dz}(e^{\ln(Az-B)}G)\,.$$

This suggests defining $\hat{G}(z,t)=e^{\ln(Az-B)}G$ for which we obtain

$$\frac{d}{dt}\hat{G}(z,t)=(z-1)(Az-B)\frac{d}{dz}\hat{G}(z,t)\,. \tag{D2}$$

This is now the same partial differential equation as Eq. (A1). Hence, the general solution for $\hat{G}$ is an arbitrary function of

$$x\equiv\left(\frac{z-1}{Az-B}\right)e^{\gamma t}\,.$$

This means, according to the definition of $\hat{G}$, that $G(z,t)$ is equal to $(Az-B)^{-1}$ times an arbitrary function of x.

Making use of the requirement

$$G(z,0)=G_0(z)$$

determines the form of the function of x. A bit of algebra establishes the full solution

$$G(z,t)=\frac{1}{1-(z-1)\bar{n}_s}\times G_0\left[\frac{1-B[(z-1)/(Az-B)]e^{\gamma t}}{1-A[(z-1)/(Az-B)]e^{\gamma t}}\right], \tag{D3}$$

in which

$$\bar{n}_s=A\gamma^{-1}(e^{\gamma t}-1)\,. \tag{D4}$$

Calculating the factorial moments from Eq. (D3) is quite tedious. The factor of $(z-1)$ in the right-hand side of Eq. (D1), coupled with the fact that the highest derivative with respect to z is first order, implies that when one operates with $(d/dz)^m$ on Eq. (D1) and then sets $z=1$, the highest derivative remaining is of order m. Thus, we may obtain differential equations, of first order in time, for the factorial moments. The coupling in the differential equation for the mth factorial moment is only to mth and lower order factorial moments, so we can successively solve for higher and higher factorial moments. For instance, the first such equation is

$$\frac{d}{dt}\bar{n}(t)=\gamma\bar{n}(t)+A\,, \tag{D5}$$

which is easily solved and yields Eq. (19).

The next order is obtained by operating twice with d/dz on Eq. (D1) and evaluating at $z=1$. The result is

$$\frac{d}{dt}G''(1,t)=2\gamma G''(1,t)+4A\bar{n}(t)\,, \tag{D6}$$

where

$$G''(1,t)=\langle n(n-1)\rangle_t=\left(\frac{d}{dz}\right)^2 G(z,t)\bigg|_{z=1}\,.$$

APPENDIX E

In this appendix, we solve for the quantum characteristic function of the unsaturated laser amplifier evolving according to Eq. (4) and for the separate effects of the stimulated and spontaneous terms identified in Eq. (11b), using Eq. (4) or Eq. (11b) and the relations

$$e^{-\zeta a^\dagger}ae^{\zeta a^\dagger}=a+\zeta$$

and

$$e^{\eta a}a^\dagger e^{-\eta a}=a^\dagger+\eta\,.$$

The partial differential equation satisfied by $C(\zeta,\eta)$, which is defined by Eq. (1), was derived by Cantrell[18]

$$\frac{\partial C(\zeta,\eta)}{\partial t}=\left[\frac{\gamma}{2}\left(\zeta\frac{\partial}{\partial\zeta}+\eta\frac{\partial}{\partial\eta}\right)+A\zeta\eta\right]C(\zeta,\eta)\,. \tag{E1}$$

In accordance with the identification of terms made by Abraham and Smith, and represented by the separation in Eq. (11b), the γ term represents stimulated emission and the remaining term represents spontaneous emission.

To solve Eq. (E1), we rearrange terms to give

$$\frac{\partial C}{\partial t}=\left[\frac{\gamma}{2}\zeta\left(\frac{\partial}{\partial\zeta}+\frac{A}{\gamma}\eta\right)+\frac{\gamma}{2}\eta\left(\frac{\partial}{\partial\eta}+\frac{A}{\gamma}\zeta\right)\right]C\,, \tag{E2}$$

which can be written

$$\frac{\partial C}{\partial t}=\left[\frac{\gamma}{2}\zeta e^{-A\zeta\eta/\gamma}\frac{\partial}{\partial\zeta}(e^{A\zeta\eta/\gamma}C)+\frac{\gamma}{2}\eta e^{-A\zeta\eta/\gamma}\frac{\partial}{\partial\eta}(e^{A\zeta\eta/\gamma}C)\right]. \tag{E3}$$

Using a new function K defined by

$$K=e^{\zeta\eta A\gamma^{-1}}(C),$$

Eq. (E3) becomes

$$\frac{\partial K}{\partial t}=\frac{\gamma}{2}\frac{\partial}{\partial(\ln\zeta)}K+\frac{\gamma}{2}\frac{\partial}{\partial(\ln\eta)}K. \tag{E4}$$

Equation (E4) has the general solution

$$K=f\left[\left(\frac{\gamma t}{2}+\ln\zeta\right),\left(\frac{\gamma t}{2}+\ln\eta\right)\right],$$

where f is an arbitrary, differentiable function of its arguments. This is equivalent to writing

$$K=g[\zeta e^{\gamma t/2},\ \eta e^{\gamma t/2}],$$

where g is another arbitrary, differentiable function. The particular function g is found by requiring that $C(\zeta,\eta,0)=C_0(\zeta,\eta)$, where C_0 describes the initial field. Using the definition of K in terms of C and the initial condition, we obtain

$$C(\zeta,\eta,t)=\exp[\zeta\eta A\gamma^{-1}(e^{\gamma t}-1)]\times C_0(\zeta e^{\gamma t/2},\ \eta e^{\gamma t/2}). \tag{E5}$$

The solution for the stimulated and spontaneous terms taken separately is simpler and proceeds along similar lines. For spontaneous emission acting alone, we have

$$\frac{\partial C}{\partial t}=A\zeta\eta C(\zeta,\eta), \tag{E6}$$

so that

$$C(\zeta,\eta,t)=e^{\zeta\eta At}C_0(\zeta,\eta). \tag{E7}$$

This describes a linearly growing Gaussian noise field of mean At superposed with the initial field.

The equation for the stimulated terms alone is

$$\frac{\partial C}{\partial t}=\frac{\gamma}{2}\left(\zeta\frac{\partial}{\partial\zeta}+\eta\frac{\partial}{\partial\eta}\right)C, \tag{E8}$$

which is of the general form of Eq. (E4) so that its solution is

$$C(\zeta,\eta,t)=C_0(\zeta e^{\gamma t/2},\ \eta e^{\gamma t/2}). \tag{E9}$$

As follows from Eq. (22) this represents the scale change in the field variables expected for linear amplification. It preserves all normalized statistical quantities.

We further note that Eq. (E5) implies for an initial vacuum ($C_0=1$) that

$$C(\zeta,\eta,t)=\exp[\zeta\eta A\gamma^{-1}(e^{\gamma t}-1)], \tag{E10}$$

which must represent amplified spontaneous emission. Clearly, the Gaussian statistics of spontaneous emission alone have been preserved. The mean

$$\bar{n}_s=A\gamma^{-1}(e^{\gamma t}-1),$$

is what would be expected from linear amplification γ and a source term A.

In conclusion, we have shown that the result of the total interaction, Eq. (E5), is the superposition of two independent fields which are the amplified spontaneous emission, Eq. (E10), and the coherently (linearly) amplified initial field, Eq. (E9).

---

*Present address: Dept. of Physics, Swarthmore College, Swarthmore, Penn. 19081.

[1]L. Mandel, Phys. Rev. A 14, 2351 (1976).

[2]F. A. Vorobev and R. I. Sokolovskii, Opt. Spektrosk. 32, 842 (1972); 36, 1066 (1974) [Opt. Spectrosc. 32, 446 (1972); 36, 625 (1974)]. See also M. Sargent, III, and M. O. Scully, in *Laser Handbook* (North-Holland, Amsterdam, 1972), edited by F. Arecchi and E. Schulz-Dubois, p. 99, where the same identification of terms is made, but without any discussion of implications.

[3]J. H. Webb, Am. J. Phys. 40, 850 (1972).

[4]M. Bertolotti, in *Photon Correlation and Light Beating Spectroscopy*, edited by H. Z. Cummins and E. R. Pike, (Plenum, New York, 1974), p. 41.

[5]M. Sargent, III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, Mass., 1974), pp. 22 and 279.

[6]A. Kastler, Proc. Int. Congr. Quantum Elec. 1, 3 (1964). See also A. Kastler, Physica 69, 330 (1973).

[7]N. B. Abraham and S. R. Smith, Pnys. Rev. A 15, 421 (1977).

[8]K. Shimoda, H. Takahashi, and C. H. Townes, J. Phys. Soc. Jpn. 12, 686 (1957).

[9]J. P. Gordon, L. R. Walker, and W. H. Louisell, Phys. Rev. 130, 806 (1963).

[10]R. Loudon, Phys. Rev. A 2, 267 (1970); and *The Quantum Theory of Light* (Clarendon, Oxford, 1973), p. 236.

[11]S. Carusotto, Phys. Rev. A 11, 1629 (1975).

[12]For an early discussion of stimulated and spontaneous emission and the source of noise, see I. R. Senitzky, Phys. Rev. 111, 3 (1958); 115, 227 (1959); 119, 1807 (1960); 123, 1525 (1961); 127, 1638 (1962); 128, 473 (1962).

[13]A. E. Glassgold and D. Holliday, Phys. Rev. 139, A1717 (1965). See also, A. E. Glassgold, in *Quantum Optics, Proceedings of the International School of Physics, "Enrico Fermi", Course XLII*, edited by R. Glauber (Academic, New York, 1967), p. 199.

[14]J. P. Gordon, in *Quantum Optics* (Ref. 13), p. 751.

[15]D. B. Scarl and S. R. Smith, Phys. Rev. A 10, 709

(1974).

[16]R. Hanbury Brown and R. Q. Twiss, Nature 177, 27 (1956); Proc. R. Soc. Lond. A 242, 300 (1957); 243, 291 (1957).

[17]N. Chandra and H. Prakash, Phys. Rev. Lett. 22, 1068 (1969); and Indian J. Pure Appl. Phys. 9, 409 (1971).

[18]This definition has also been made in Ref. 11 and by C. D. Cantrell, Princeton Univ. Tech. Rep. No. 11, Contract AT(30-1)-3406 (1971) (unpublished). Similar characteristic functions were considered in Refs. 9 and 13.

[19]The characteristic function for the superposition (addition) of two independent random variables is well known to factor into the product of the characteristic functions of the two fields separately. See, for example, Ju. V. Linnik and I. V. Ostrovskii, *Decomposition of Random Variables and Vectors*, translated from Russian (American Mathematical Society, Providence, R.I., 1977), especially Chap. 2, or the references cited in footnote 27.

[20]Excess correlations in such resonance fluorescence problems have been investigated in detail recently. See, for example, H. J. Kimble and L. Mandel, Phys. Rev. A 13, 2123 (1976).

[21]It was in this approximation that Mandel in Ref. 1 argued that stimulated emission was a cause of correlations. For the correspondence between this approximation and the laser amplifier, see the discussion following Eq. (29).

[22]Reference 5, Chap. 16.

[23]Reference 5, Chap. 17.

[24]R. Glauber, Phys. Rev. 131, 2766 (1963). For an interesting discussion of such a superposition in terms of wave packets, see R. Glauber, in *Quantum Optics, Proceedings of Scottish University Summer School, 10th, Edinburgh, 1969*, edited by S. M. Kay and A. Maitland (Academic, New York, 1970), p. 114. This contrasts with another wave-packet model reported by C. Benard, Phys. Lett. 41A, 191 (1972).

[25]R. Glauber, in *Quantum Optics* (Ref. 13), p. 46; or J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968), pp. 21–26.

[26]The generating function for the sum of two independent variables does factor into the product of the two separate generating functions. Since we are considering the photon number generating function, the addition of two independent fields leads to intensity interference effects which prevent factoring. See reference in footnote 19.

[27]A Kolomogorov equation is a first-order time evolution equation of the form

$$\frac{d}{dt} P_n = \Sigma_m \lambda_{mn} P_m ,$$

where

$$\Sigma_n \lambda_{mn} = 0$$

and

$$\lambda_{nn} < 0 .$$

Such an equation governs a Markoff process [see A. B. Clark and R. L. Disney, *Probability and Random Processes for Engineers and Scientists* (Wiley, New York, 1970), p. 281; and W. Feller, *An Introduction to Probability Theory and its Applications*, 3rd ed. (Wiley, New York, 1968), Vol. I, Chap. XVII]. These are exactly the conditions for the time evolution of a density matrix for which the first condition conserves probability while the second guarantees that $0 \leqslant \rho_n \leqslant 1$ is maintained for all time.

[28]Actually, in Appendix B, we show how one might obtain a BE distribution from Eqs. (8b) and (9) in the limit as one adds noninterfering photons created during all infinitesimal time (or distance) intervals in the past. However, no physical significance can be given to this exercise.

[29]The separation in Eq. (11a) and (11b) appears without discussion in a paper by W. H. Louisell [in *Quantum Optics* (Ref. 24), p. 191].

[30]It should be noted that all other equations for $\dot{\rho}_n$ considered in this paper are valid Kolmogorov equations.

[31]B. Picinbono and M. Rousseau, Phys. Rev. A 15, 1648 (1977).

[32]V. Y. Anisimov and B. Sotskii, Opt. Spektrosk. 39, 781 (1975) [Opt. Spectrosc. 39, 441 (1975)].

[33]B. Picinbono, Phys. Rev. A 16, 449 (1977).

[34]These results and conclusions were also reached in Ref. 10 by another method of obtaining differential equations for the moments.

[35]We note that Eq. (17) is the product of Eq. (12b) [or similarly, Eq. (B6)] and Eq. (9), but insist that no physical significance can be derived from this fact.

[36]Note that the diffusion term in two dimensions,

$$D\left(\frac{\partial^2}{\partial \alpha_r^2} + \frac{\partial^2}{\partial \alpha_i^2}\right) P = D \nabla_\alpha{}^2 P ,$$

is equivalent to the diffusion term in the Fokker-Planck equation $4D\, \partial^2 P / \partial \alpha \partial \alpha^*$ identified in Refs. 7 and 17 as corresponding to spontaneous emission.

[37]S. Chandrasekhar, Rev. Mod. Phys. 15, 1 (1943). It is pointed out in the bibliographical notes of this reference that Lord Rayleigh first considered the problem of random walk in a plane in a context quite similar to the present one.

[38]Reference 5, p. 236.

[39]See, for example, the references in footnote 27.