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Amplified-spontaneous-emission intensity fluctuations

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Intensity fluctuation characteristics of amplified-spontaneous emission are found using intensity rate equations and the noise-amplifier approximation for saturation coupling between two and four competing modes. The existence of more than one mode leads to the retention of relatively large fluctuations in each mode even as the amplifier gain is saturated, an operating regime which by contrast has been shown to reduce significantly the fluctuations when only a single beam is present. The noise-amplifier approximation is found to be an adequate approximation of the effects of the spontaneous emission distributed along the length of a laser amplifier. The intensity fluctuations in all cases approach limiting distribution functions as the amplifier becomes heavily saturated. Characteristics of these limiting distributions are found for one, two, and four beams; for combinations of beams copropagating and counterpropagating; in the presence and absence of loss; for differences in gain; and for saturation of the sources due to high intensities.

I. INTRODUCTION

Early theories of amplified-spontaneous emission (ASE) considered only the growth of the average intensity or variation of the spectral density with increasing amplifier length, or variations within an amplifier of fixed length.[1] For unsaturated systems, each mode could be considered independently. In even the simplest theories, however, gain saturation couples the independent spectral, spatial, or polarization modes of the system through their interaction with a common atomic population.

Recognizing this, more recent theories of ASE intensity growth and spectral-width variation have given close attention to models which assume the existence of signals propagating in both directions in one-dimensional amplifying media.[2–6] Interestingly enough, however, one can see that the results published for these bidirectional theories are not noticeably different from the single-mode results.[1,5] In our examination of multimode-rate-equation solutions, we will demonstrate why they so closely agree with the single-mode results.

In addition to bidirectionality, there is the added dimension of two polarization modes propagating in each direction.[7] In some systems such as discharge-excited gas-laser amplifiers, the excited atoms are randomly polarized and thus the two polarization states of the field couple equally strongly to the atoms and can have equal influence on the gain saturation by depletion of the number of excited atoms. In other systems, such as in transversely (optically) pumped dye-laser systems, the molecules are polarized and the coupling between the two optical polarizations is significantly reduced.[8] Some studies of the variations due to different coupling strengths, resulting from detuning of competing modes in lasers, have been reported.[9,10] As ASE generally involves broadband-optical signals, no such investigation is envisioned for the present work.

While recognizing the many complications which one might introduce in studies of ASE, we will limit the scope of the present work to homogeneously broadened systems. Since, in this case, the full broadband-optical signal interacts with each atom and the signal (after gain narrowing) fluctuates slowly with respect to the polarization dephasing time, the growth of the total intensity may be considered to be governed by an intensity rate equation.[5–7] Limitations of this assumption have been discussed in detail by Hopf.[11] Ultimately, the model is limited by the onset of "cross-spectral coupling effects" first discussed by Gamo[12] and later identified more thoroughly in the theoretical work of Hopf[11] and that of Menegozzi and Lamb.[13] When the intensities become sufficiently strong, relative to the saturation intensity, the finite bandwidth of the signal comes into play. The so-called randon-phase approximation no longer describes the atomic response to the separate optical-frequency components and the rate-equation methods are no longer valid.

Systematic studies, either experimentally or theoretically, of intensity fluctuations of ASE sources have only recently been made. Hopf's[11] single-mode theory predicted that saturation would quickly reduce the initial thermal fluctuations of the spontaneous emission. For example, the normalized variance of the fluctuations is shown to go to zero at long amplifier lengths, assuming continued validity of the intensity rate equation for very high intensities.

Experimental studies of the intensity fluctuations of cw ASE have been reported for the 3.51-μm line of xenon in helium-xenon gas discharges and the 3.39-μm line of neon in helium-neon gas discharges.[14-17] Intensity fluctuation studies of pulsed ASE systems have considered radiation at 540-nm in neon[18] and 694-nm in ruby.[19] Cosmic maser action (presumably ASE) has been observed at microwave frequencies for CO and OH.[20,21]

For these systems the approximations of homogeneous broadening and intensity rate equations offer only a modestly acceptable approximation. For example, in helium-neon the reported minimum spectral width of 17 MHz compares to a pressure-broadened homogeneous linewidth of 146 MHz, while in helium-xenon the reported minimum spectral width of 30 MHz compares to a pressure-broadened homogeneous linewidth of 85 MHz. Allen and Peters[3] have discussed the evidence for at least fivefold narrowing in the mainly homogeneously broadened cosmic-maser lines.

The truly approximate nature of the assumptions of this paper in their application to ASE systems is evident. The results apply much more rigorously to multimode-laser systems where the individual modes have very narrow linewidths. (See Refs. 9, 10, and 22). However, the experimental results, as discussed below, show remarkable qualitative consistency with our theoretical results suggesting sufficient roubustness of the theory to justify the drawing of physical insight from our model.

Studies of the characteristics of cosmic masers have left a particular puzzle. The sources are known to have extreme brightness, with evidence of spectral narrowing. Despite the expectation of heavy gain saturation neither spectral rebroadening nor departures from non-Gaussian statistics have been observed.[20,21,3] These results have been attributed to the interaction of many modes,[10,19,21,23] and the intensity-rate-equation model developed here demonstrates how readily such interactions limit the effect of gain saturation on the intensity fluctuations of a single mode.

Early measurements by Gamo,[24,14,15] using cascaded amplifier chains separated by Faraday rotation isolators to achieve quasiunidirectional systems, displayed a leveling out of the normalized variance at a nonzero, through reduced, value. The nonzero limiting value did not agree with the single-mode (intensity-rate-equation) predictions, nor could it readily be explained in terms of cross spectral effects, although saturation was clearly evident.

We have recently reported a theory of intensity fluctuations in bidirectional ASE sources which showed that the coupling between two oppositely propagating beams would produce a nonzero limiting value for the normalized variance of each beam.[25] Even more recently, our experimental measurements on xenon-helium systems similar to Gamo's but using only a single variable-length discharge have demonstrated the leveling out of the normalized variance as the length is increased beyond that necessary to cause saturation of the gain.[16,17] We have also clearly demonstrated that this leveling out of the normalized variance is correlated with the onset of anticorrelations in the fluctuations of beams of different linear polarizations propagating in the same direction.[26] The general agreement of these experimental results and the early theories of coupled intensity-rate equations leads us to the more complete theoretical treatment presented in this paper.

II. GENERAL APPROACH

We describe the variation along the amplifier of several coupled modes by rate equations of the form

$$\frac{dx_\mu}{dz}=\epsilon_\mu\left\{\frac{g_\mu x_\mu}{\left[1+\sum_\nu \xi_{\mu\nu}x_\nu\right]}-\alpha_\mu x_\mu+\frac{g_\mu x_{0\mu}}{\left[1+\sum_\nu \xi'_{\mu\nu}x_\nu\right]}\right\}, \qquad (2.1)$$

where $\epsilon_\mu = 1$ for signals propagating in the $+z$ direction and $\epsilon_\mu = -1$ for counterpropagating signals, g_μ is the small signal gain for the μth mode, x_μ is the normalized intensity, α_μ is a linear-loss coefficient due to diffraction or scattering, $g_\mu x_{0\mu}$ represents the distributed spontaneous-emission source, and $\xi_{\mu\nu}$ and $\xi'_{\mu\nu}$ are coupling constants indicating the relative strength of the saturation depletion of the gain and spontaneous emission, respectively, for the μth mode due to the νth intensity. The incremental gain

$$G_\mu = g_\mu \left[1 + \sum_\nu \xi_{\mu\nu} x_\mu \right]^{-1} , \tag{2.2}$$

is proportional to the population inversion (N_2-N_1), while the spontaneous emission

$$X_{0\mu} = g_\mu x_{0\mu} \left[1 + \sum_\nu \xi'_{\mu\nu} x_\nu \right]^{-1} \tag{2.3}$$

depends on the upper-state population N_2, alone. Thus, in general, the saturation of G_μ and $X_{0\mu}$ will differ leading to $\xi'_{\mu\nu} \neq \xi_{\mu\nu}$.

We will first analyze these equations in the approximation that they describe the evolution of the mean intensity, gaining insight into the implications of different values for the key parameters. The deviations of this mean-intensity rate-equation approach from the more exact approach of averaging the result over the intensity fluctuations has previously been shown to be small.[25]

To calculate the intensity fluctuations, we will use the so-called "noise-amplifier" approximation which has been used previously.[11,13,25,27] Instead of distributed spontaneous emission, the source is taken to be a statistically varying input signal. The intensity rate equations are treated as stochastic differential equations and the output fluctuations are determined by averaging over the input fluctuations. The "noise-input" signals are taken to have the negative-exponential intensity-probability-distribution functions characteristic of linearly amplified spontaneous-emission noise.

Our general goal is to describe differences among single-mode (polarized) signals in single-beam, bidirectional, and two-polarization bidirectional systems. Even more modes can easily be treated by our theoretical approach, but we assume that the two-polarization bidirectional case may be the most physically meaningful in describing the typical experimental conditions in a long cylindrical homogeneously broadened amplifer.

For calculational simplicity, we will make the assumption that all modes have the same coupling constants $\xi_{\mu\nu} = 1$. Additionally, while most interest will be concentrated on the case when all modes have the same gain $g_\mu = g_\nu = g$, we will indicate the nature of the solutions when the gain constants are not equal. Since, also quite generally, experimental systems have relatively high gain and little loss, we will take $\alpha_\mu = 0$ for our most detailed studies. When $\alpha_\mu \neq 0$, the most interesting alteration of the results is a change in the asymptotic form of the fluctuations which will be displayed.

III. MEAN-INTENSITY GROWTH

A. Saturated source (four-level laser)

In systems where $N_1 = 0$, the gain and spontaneous emission will have the same saturation behavior ($\xi'_{\mu\nu} = \xi_{\mu\nu}$). For a single mode, the equation

$$\frac{dx}{dz} = \frac{g(x + x_0)}{1 + x} \tag{3.1}$$

has solution for $x(L)$ (the ASE output of an amplifier of length L) given by the implicit function

$$(1 - x_0) \ln \left[\frac{x(L)}{x_0} + 1 \right] + x(L) = gL . \tag{3.2}$$

Considering two modes having the same gain and propagating in the same direction, we observe that in calculations involving only mean intensities, $x_1(z) = x_2(z)$, so that the equations

$$\frac{dx^{1,2}}{dz} = \frac{g(x_{1,2} + x_0)}{1 + x_1 + x_2} \tag{3.3}$$

reduce to the form

$$\frac{dx_1}{dz} = \frac{g(x_1 + x_0)}{1 + 2x_1} , \tag{3.4}$$

which has clear solution

$$(1 - 2x_0) \ln \left[\frac{x_1(L)}{x_0} + 1 \right] + 2x_1(L) = gL . \tag{3.5}$$

The effect of coupling two modes in this fashion is the same functional form of intensity-output dependence on amplifier length as in the single-mode case with twice the effective saturation parameter (to which the intensities were previously normalized). In this case, it is also true that the beam varies along the length of the amplifier, as if the saturation parameter were doubled at each point.

The case of two counterpropagating modes as governed by the equation

$$\frac{dx^{\pm}}{dz}=\pm g\frac{(x^{\pm}+x_0)}{1+x^{+}+x^{-}} \tag{3.6}$$

has been considered by Casperson[5] and is readily shown to have solution

$$(1-2x_0)\ln\left[\frac{x^{+}(L)}{x_0}+1\right]+2x^{+}(L)=gL\ . \tag{3.7}$$

This is quite a remarkable result. The output intensity of one of two counterpropagating beams is identical to the output of one of two copropagating signals in amplifiers having the same length and the same unsaturated spontaneous emission along the amplifier. The output in the bidirectional case is again the same as the output for a single mode with twice the saturation parameter; however, in this case it does not grow along the amplifier length as if the saturation intensity were doubled at each point. Thus, the intensities in the two cases vary differently along the length of the amplifier when saturation is important. The copropagating beams have source values of gx_0 at $z=0$ while the counterpropagating beams have source values of $gx_0[1+x^{+}(L)]^{-1}$ at their respective starting ends. Correspondingly, the slopes of the curves $x(z)$ evaluated at $z=L$, differ. In the copropagating case,

$$\frac{dx_1(L)}{dz}=\frac{g[x_1(L)+x_0]}{1+2x_1(L)}\ , \tag{3.8}$$

while in the counterpropagating case,

$$\frac{dx^{+}(L)}{dz}=\frac{g[x^{+}(L)+x_0]}{1+2x_1(L)}\ , \tag{3.9}$$

where from Eqs. (3.5) and (3.7), we see that $x^{+}(L)=x_1(L)$.

The fact that the output intensity for the bidirectional and unidirectional cases differ only by a factor of 2 in the values of the saturation parameter, explains the essential agreement of earlier spectral line-shape studies in the two cases.[1,5] The general result was that variation of the spectral line shapes with amplifier length was demonstrably insensitive to even an order-of-magnitude variation in the saturation parameter. Thus the saturation-induced rebroadening in one- and two-beam theories should be essentially the same.

The solution for one of four similar modes coupled in this fashion (which, in particular, includes the case of two beams propagating in each direction) is readily shown to be

$$(1-4x_0)\ln\left[\frac{x_1^{+}(L)}{x_0}+1\right]+4x_1^{+}(L)=gL\ . \tag{3.10}$$

Again this result is functionally equivalent to the earlier results, requiring only modification of a single parameter to make the results indistinguishable. Thus, measurements of intensities will be relatively insensitive in providing guidance in the selection of, or verification of, multibeam models. Such selection could occur only if the saturation parameter were determined by an independent measurement, perhaps by the determination of single-mode-gain characteristics from studies of amplification of an external source. With this value in hand, one might then use the ASE intensity data to determine the number of beams present, and the strength of the interaction.

B. Unsaturated source (three-level laser)

The functional equivalence of the copropagating and counterpropagating cases considered in Sec. III A above is model dependent as evidenced by consideration of an alternative rate equation appropriate when $(N_2\text{-}N_1)<<N_2$:

$$\frac{dx_\mu}{dz}=\epsilon_\mu\left[\frac{g_\mu x_\mu}{\left[1+\sum_\nu x_\nu\right]}+g_\mu x_0\right]\ . \tag{3.11}$$

For a single mode, the solution is given by

$$(1-x_0)\ln\left[\frac{x(L)}{x_0}+1\right]+x(L)=g(1+x_0)L\ . \tag{3.12}$$

The difference between this unsaturated-source model and the previously considered saturated-source model is small. Equation (3.12) is just Eq. (3.2) with a change of parameters g to $g(1+x_0)$. Since the spontaneous-emission source is much less than the saturation intensity in most systems, this correction is insignificant and the two models are practically indistinguishable in their results.

Considering two copropagating modes, it is readily apparent for the same reasons as in Sec. III A that the result is

$$(1-2x_0)\ln\left[\frac{x(L)}{x}+1\right]+2x_1(L)=g(1+x_0)L\ . \tag{3.13}$$

The case of two counterpropagating modes has been solved previously[6] and is given by the equation

$$\frac{2}{(1+2x_0)^{3/2}}\ln\frac{2x_0+[2x_0+1+(2x_0+1)^{1/2}]x^+(L)}{2(x_0^2+x_0(2x_0+1)\{\frac{1}{2}[x^+(L)]^2+x^+(L)\})^{1/2}}+\frac{2x^+(L)}{1+2x_0}=gL\ . \tag{3.14}$$

Neglecting terms of order x_0 compared to 1, we can compare these two results more easily as follows: two copropagating beams

$$\ln\left[\frac{x_1(L)}{x_0}+1\right]+2x_1(1)=gL\ , \tag{3.15}$$

two counterpropagating beams

$$\ln\left[\left[\frac{x^+(L)}{x_0}+1\right]\left[\frac{x^+(L)+x_0}{x^+(L)+x_0+\frac{1}{2}[x^+(L)]^2}\right]\right]+2x^+(L)=gL\ . \tag{3.16}$$

The results for $x(L)$ are the same for both $x(L)>>1$ and $x(L)<<1$, and in these limits agree exactly with the results for two beams treated in the model considered in Sec. III A. There are noticeable deviations in the vicinity of the onset of saturation $[x(L)\approx 1]$ but these can be shown to never exceed 4%. Thus, although the exact equivalence is broken, the strong similarity and essential equivalence between the copropagating and counterpropagating models will prevent a qualitative distinction being made in evaluating experimental data.

An important general result is that the output of a single mode for a long-length amplifier depends on the magnitude of the spontaneous emission which acts as an input source for an amplifier without noise. We are thus reasonably justified in considering the noise-amplifier approximation in our later studies.

IV. OUTPUT INTENSITY FOR THE GENERAL LOSSLESS NOISE AMPLIFIER

For the lossless noise amplifier, Eq. (2.1) becomes

$$\frac{dx_\mu}{dz}=\epsilon_\mu\frac{g_\mu x_\mu}{\left[1+\sum_\nu x_\nu\right]}\ . \tag{4.1}$$

Following Gray and Casperson[7] who solved the particular case of two copropagating signals, we will solve the general case of coupled-copropagating and counterpropagating signals by observing the particular relations that permit us to write the equation for x_ν in terms of the sources $x_{\nu i}$, thereby completing the uncoupling of the equations. Here, $x_{\nu i}$ denotes the input signal in the νth mode, which is used in place of distributed spontaneous emission.

A. Relations among two copropagating beams

Taking $\epsilon_\mu=1$, we have the following relation when $\epsilon_\nu=1$:

$$\frac{1}{x_\mu g_\mu}\frac{dx_\mu}{dz}=\left[1+\sum_\lambda x_\lambda\right]^{-1}=\frac{1}{x_\nu g_\nu}\frac{dx_\nu}{dz}\ . \tag{4.2}$$

Thus, there exist constants along the amplifier

$$d_{\mu\nu}=[x_\mu(z)]^{1/g_\mu}[x_\nu(z)]^{-1/g_\nu}\ . \tag{4.3}$$

These constants represent the physical fact that the number of gain lengths for the two beams always differ by a factor equal to the ratio of the small signal gain constants

$$\frac{x_\mu(z)}{x_\nu(z)}=\frac{x_{\mu i}\exp\left[\int_0^z G_\mu dz\right]}{x_{\nu i}\exp\left[\int_0^z G_\nu dz\right]}=\frac{x_{\mu i}}{x_{\nu i}}\exp\left[\left[\int_0^z G_\mu dz\right]\left[1-\frac{g_\nu}{g_\mu}\right]\right]\ . \tag{4.4}$$

The constants may be evaluated at the common input position of the two signals ($z=0$) yielding

$$d_{\mu\nu}=(x_{\mu i})^{1/g_\mu}(x_{\nu i})^{-1/g_\nu} . \quad (4.5)$$

Combining Eqs. (4.3) and (4.5), we can then substitute for $x_\nu(z)$ in Eq. (4.1) in terms of $x_\mu(z)$ and the input signals $x_{\mu i}$ and $x_{\nu i}$.

B. Relations between x_μ and counterpropagating beams x_λ

In this case, $\epsilon_\mu=1$ and $\epsilon_\gamma=-1$, and

$$\frac{1}{x_\mu g_\mu}\frac{dx_\mu}{dz}=\left[1+\sum_\lambda x_\lambda\right]^{-1}=\frac{1}{x_\gamma g_\gamma}\frac{dx_\gamma}{dz} , \quad (4.6)$$

so that there again exist constants along the amplifier given by

$$d_{\mu\gamma}=[x_\mu(z)]^{1/g_\mu}[x_\gamma(z)]^{1/g_\gamma} , \quad (4.7)$$

which can be evaluated at the output of the μth beam giving

$$d_{\mu\gamma}=(x_{\mu o})^{1/g_\mu}(x_{\gamma i})^{1/g_\gamma} , \quad (4.8)$$

where $x_{\mu o}$ is the output of the μth beam $[x_\mu(L)]$, and since the output end for the μth beam is the input for the γ beams.

C. General solution completed

Using the $d_{\mu\nu}$ and $d_{\mu\gamma}$, Eq. (4.1) may be rewritten

$$\frac{dx_\mu}{dz}=g_\mu x_\mu\left[1+x_\mu+\sum_{\substack{\nu\neq\mu\\ \epsilon_\nu=1}}(d_{\mu\nu})^{-g_\nu}x_\mu^{g_\nu/g_\mu}+\sum_{\substack{\lambda\neq\mu\\ \epsilon_\lambda=-1}}d_{\mu\lambda}^{g_\lambda}x_\mu^{-g_\lambda/g_\mu}\right]^{-1} . \quad (4.9)$$

Rearranging this equation yields

$$g_\mu dz=\frac{dx_\mu}{x_\mu}\left[1+x_\mu+\sum_{\substack{\nu\neq\mu\\ \epsilon_\nu=1}}(d_{\mu\nu})^{-g_\nu}x_\mu^{g_\nu/g_\mu}+\sum_{\substack{\lambda\neq\mu\\ \epsilon_\lambda=-1}}d_{\mu\lambda}^{g_\lambda}x_\mu^{-g_\lambda/g_\mu}\right] , \quad (4.10)$$

which can be integrated along the full length of the amplifier with the result

$$g_\mu L=\ln\left[\frac{x_{\mu o}}{x_{\mu i}}\right]+(x_{\mu o}-x_{\mu i})+\sum_{\substack{\nu\neq\mu\\ \epsilon_\nu=1}}\frac{g_\mu}{g_\nu}d_{\mu\nu}^{-g_\nu}(x_{\mu o}^{g_\nu/g_\mu}-x_{\mu i}^{g_\nu/g_\mu})+\sum_{\substack{\lambda\neq\mu\\ \epsilon_\lambda=-1}}\frac{g_\mu}{g_\lambda}d_{\mu\nu}^{g_\lambda}(x_{\mu o}^{-g_\lambda/g_\mu}-x_{\mu i}^{-g_\lambda/g_\mu}) . \quad (4.11)$$

Substituting for the d's, from Eqs. (4.5) and (4.8), we find

$$g_\mu L=\ln\left[\frac{x_{\mu o}}{x_{\mu i}}\right]+(x_{\mu o}-x_{\mu i})+\sum_{\substack{\nu\neq\mu\\ \epsilon_\nu=1}}\frac{g_\mu}{g_\nu}\left\{x_{\nu i}\left[\left[\frac{x_{\mu o}}{x_{\mu i}}\right]^{g_\nu/g_\mu}-1\right]\right\}-\sum_{\substack{\lambda\neq\mu\\ \epsilon_\lambda=-1}}\frac{g_\mu}{g_\lambda}\left\{x_{\lambda i}\left[1-\left[\frac{x_{\mu o}}{x_{\mu i}}\right]^{g_\lambda/g_\mu}\right]\right\} , \quad (4.12)$$

which may be simplified to the form

$$g_\mu L=\ln\left[\frac{x_{\mu o}}{x_{\mu i}}\right]+\sum_\nu\frac{g_\mu}{g_\nu}x_{\nu i}\left[\left[\frac{x_{\mu o}}{x_{\mu i}}\right]^{g_\nu/g_\mu}-1\right] . \quad (4.13)$$

This final form explicitly displays the fact that the distinction among the copropagating and counterpropagating beams was not necessary. In the lossless amplifier they contribute exactly the same influence on the evolution of one particular beam. The implicit function in Eq. (4.13) may now be solved when the initial conditions $x_{\nu i}$ are specified for all ν.

V. CALCULATION OF FLUCTUATIONS

Assuming that our coupled intensity rate equations describe a stochastic process, we may determine the fluctuations in the output of a single beam by averaging the result over the distributions describing the fluctuations of the statistically independent input signals. For example, the moments of the output-intensity-probability-distribution functions are given by

$$\int (x_{\mu o})^n P(x_{\mu o}) dx_{\mu o} = \int [x_{\mu o}(L,\{x_{\nu i}\})]^n \prod_\nu [P_\nu(x_{\nu i}) dx_{\nu i}] . \quad (5.1)$$

We will calculate below the fluctuation characteristics of limiting cases (long, heavily saturated amplifiers) and the general length dependence for a variety of values of the parameters. While this method is generally applicable to arbitrarily chosen input-fluctuation characteristics, for ASE the fluctuations will be assumed to have negative-exponential intensity-probabilty-distribution functions characteristic of spontaneous-emission noise:

$$P_\nu(x_{\nu i}) = \langle x_{\nu i}\rangle^{-1} \exp(-x_{\nu i}\langle x_{\nu i}\rangle^{-1}) . \quad (5.2)$$

where the integrations in Eq. (5.1) cannot be done in closed form, we can perform the integration by Gauss-Laguerre[28] quadratures. A modified binary search is used to find the value of the implicit function for each set of input values. We use the Cartesian direct-product formula to accomplish the multivariate Gauss-Laguerre integration

$$\int\int F(y,z) e^{-y} e^{-y} dy\, dz = \sum_{i,j}^{M} F(x_i,y_i) w_i w_j , \quad (5.3)$$

where the roots (x_i) and weights (w_i) are tabulated for various values of M.[29] For our final calculations, M was chosen large enough to give 1% or better accuracy as determined by extrapolation from results for smaller values of M. This approximate accuracy was confirmed by comparison with the analytic formulas for the asymptotic values when available.

Specific characterization of the fluctuations of the output signals will include the normalized cumulants[30] K_n of the distributions of a single-beam and cross-correlation coefficients describing correlations in the fluctuations of two beams. For reference, the normalized cumulants for a negative exponential distribution have values $K_n = (n-1)!$ for $n > 2$, while $K_n = 0$ for a constant intensity distribution. Commonly used formulas include the normalized variance

$$K_2 = (\langle x_{\mu o}^2\rangle - \langle x_{\mu o}\rangle^2)(\langle x_{\mu o}\rangle^{-2}) , \quad (5.4)$$

and cross-correlation coefficients

$$C_{\mu\nu} = (\langle x_{\mu o} x_{\nu o}\rangle - \langle x_{\mu o}\rangle\langle x_{\nu o}\rangle)(\langle x_{\mu o}\rangle\langle x_{\nu o}\rangle)^{-1} , \quad (5.5)$$

which measures the correlations in terms of the mean intensities, and

$$C'_{\mu\nu} = (\langle x_{\mu o} x_{\nu o}\rangle - \langle x_{\mu o}\rangle\langle x_{\nu o}\rangle)[\langle(\Delta x_{\mu o})^2\rangle\langle(\Delta x_{\nu o})^2\rangle]^{-1/2} , \quad (5.6)$$

which measures the correlations as a fraction of the mean fluctuations of the two signals.

VI. LIMITING RESULTS FOR LONG AMPLIFIER LENGTHS (HEAVILY SATURATED AMPLIFIERS)

A. Lossless amplifiers

For very long lengths, the intensities will be much larger than the input-source terms, so Eq. (4.13) simplifies with the result

$$g_\mu L = \sum_\nu \frac{g_\mu}{g_\nu} x_{\nu i} \left[\frac{x_{\mu o}}{x_{\mu i}}\right]^{g_\nu/g_\mu} , \quad (6.1)$$

which has a particularly simple explicit form when all of the g_ν's are equal to g:

$$x_{\mu o} = gL \left[1 + (x_{\mu i})^{-1} \sum_{\nu\neq\mu} x_{\nu i}\right]^{-1} . \quad (6.2)$$

In this case the calculations are simplified still further when all of the $x_{\nu i}$ have the same mean intensity $\langle x_i\rangle$, as

$$P\left[\sum_{\nu\neq\mu} x_{\nu i}\right] \equiv P(I) = \langle x_i\rangle^{-(N+1)} \frac{I^{N-2}}{(N-2)!} \exp[-I\langle x_i\rangle^{-1}] , \quad (6.3)$$

where there are N terms in the sum. The solutions of Eq. (5.1), using Eq. (6.2) for $x_{\mu o}$, are just solutions to a two-beam problem with one source of the form Eq. (6.3) and the other of the form Eq. (5.2). The integrations can be completed in closed form with the results shown in Table I.

When there are two beams with different gain coefficients, Eq. (6.1) becomes

$$g_1 L = x_{1o} + \frac{g_1}{g_2} x_{2i} \left[\frac{x_{1o}}{x_{1i}}\right]^{g_2/g_1} . \quad (6.4)$$

Results are shown in Table II. These can be com-

TABLE I. Limiting characteristics of the intensity fluctuations (equal gain) (NC—not calculated).

	$x_{\mu o}$	K_2	K_3	K_4	K_5	K_6	K_7	C_{12}	C'_{12}
Unsaturated source									
N beams without loss (copropagating or counterpropagating)	gL/N	$\frac{N-1}{N+1}$	NC	NC	NC	NC	NC	$-\frac{1}{N+1}$	$-\frac{1}{N-1}$
Two beams									
Copropagating									
with loss,	$gL/2$	1/3	0	−2/15	0	0.25	0	−1/3	−1
without loss	$gL/2$	1/3	0	−2/15	0	0.25	0	−1/3	−1
Counterpropagating									
without loss,	$gL/2$	1/3	0	−2/15	0	0.25	0	−1/3	−1
with loss	$gL/1.4$	0.28	−0.096	−0.1	0.18	0.09	−0.67	−0.17	−0.62
Four beams									
Copropagating									
without loss,	$gL/4$	3/5	0.43	0.015	−0.83—	1.7	1.4	−1/5	−1/3
with loss	$gL/4$	3/5	0.43	0.015	−0.83	−1.7	1.4	−1/5	−1/3
Two in each direction									
without loss,	$gL/4$	3/5	0.43	0.015	−0.83	−1.7	1.4	−1/5	−1/3
with loss	$gL/2.6$	0.56	−0.014	−0.34	−0.76	0.57	5.1	−0.21	−0.37
(countercorrelations)								−0.089	−0.16
Saturated source									
Two beams									
Copropagating									
with loss	unchanged from unsaturated two-beam results								
without loss	unchanged from unsaturated two-beam results								
Counterpropagating									
without loss,	unchanged from unsaturated two-beam results								
with loss	$gL/2$	0.56	0	−0.51	0	2.0	0	−0.57	−1

TABLE II. Long-length limiting characteristics of the intensity fluctuations in a lossless two-beam amplifier (unequal gain) (NC—not calculated).

		Normalized cumulants of the distribution of x_{1o}						Correlations	
$\frac{g_1}{g_2}$	$\frac{\langle x_{1o}\rangle}{g_1L}$	K_2	K_3	K_4	K_5	K_6	K_7	C_{12}	C'_{12}
0.2	1×10^{-7}	1.136	2.934	12.66	78.5	6.4×10^{2}	6.4×10^{3}	NC	NC
0.5	7.2×10^{-5}	2.065	12.38	121.8	1.6×10^{2}	2.5×10^{4}	4.9×10^{5}	-1.45×10^{-4}	−0.998
0.9	0.23	1.006	1.333	0.98	NC	NC	NC	−0.289	−1
0.99	0.47	0.376	0.0304	−0.167	NC	NC	NC	−0.332	−1
1	0.5	$0.3\bar{3}$	0	2/15	0	0.25	0	−1/3	−1
1.01	0.532	0.29	−0.021	−0.1	3.5×10^{-2}	0.16	−0.13	−0.332	−1
1.1	0.777	0.083	−0.032	6.7×10^{-3}	7.3×10^{-3}	−0.012	5.9×10^{-3}	−0.289	−1
2	0.9999	0	0	0	0	0	0	-0.2×10^{-4}	−0.998
5	1	0	0	0	0	0	0	NC	NC

pared to recent studies of gain variations in a multimode laser.[22]

Note specifically that for one beam $K_2=0$ and the C's are undefined. This is the result alluded to earlier that saturation reduces the fluctuations when only a single beam is present until no fluctuations remain. The output is a constant intensity.

The general results show, in agreement with earlier results, that the presence of more than two beams leads to a sustaining of the variance of the fluctuations at a value closer to that characteristic of thermal light. We have shown previously that the two-beam case has a limiting intensity-probability-distribution function which is constant from zero to gL.

Initial negative exponential fluctuations are retained in the limit as $N\rightarrow\infty$, as has been suggested as an explanation for the thermal fluctuations observed in heavily saturated cosmic masers.[23,20]

The close agreement with the theories of coupled modes in a laser[9,10] is not surprising. However, it indicates that the use of electric-field-amplitude Langevin rate equations provides little additional insight into the characteristics of the intensity fluctuations. The intensity rate equations are sufficient since in either case the coupling occurs because of saturation effects involving the addition of the intensities rather than the field amplitudes.

B. Amplifiers with loss

If one keeps the loss in the intensity rate equations then the limiting output is independent of the amplifier length. The maximum possible output of any beam is $x_{\mu o}=\alpha_\mu^{-1}(g_\mu-\alpha_\mu)=K_\mu$. Assuming that the gain and loss are the same for all of the interacting beams, we can find the limiting results for copropagating and counterpropagating cases.

For two beams the limiting intensities are given by

$$x_{1o}=\frac{x}{1+x_{2i}/x_{1i}} \tag{6.5}$$

in the copropagating case, and by

$$x_{1o}=\begin{cases}(K-x_{1i}), & x_{1i}\geq x_{2i}\\ \dfrac{x_{1i}}{x_{2i}}(K-x_{2i}), & x_{2i}\geq x_{1i}\end{cases} \tag{6.6}$$

in the counterpropagating case.[25] For four beams, the result for all four propagating in the same direction is

$$x_{1o}=K\left[1+\frac{1}{x_{1i}}(x_{2i}+x_{3i}+x_{4i})\right]^{-1}, \tag{6.7}$$

while when there are two beams traveling in each direction the result is

$$x_{1o}^{+}=\begin{cases}[K-(x_{1i}^{-}+x_{2i}^{-})]\left[\left[1+\dfrac{x_{2i}^{+}}{x_{1i}^{+}}\right]\right]^{-1}, & x_{2i}^{+}+x_{1i}^{+}\geq x_{2i}^{-}+x_{1i}^{-}\\ x_{1i}[K-(x_{1i}^{+}+x_{2i}^{+})](x_{2i}^{+}+x_{1i}^{-})^{-1}, & x_{2i}^{+}+x_{1i}^{+}\leq x_{2i}^{-}+x_{1i}^{-}\end{cases} \tag{6.8}$$

The general result with loss quickly becomes apparent. If the beams are propagating together, the intensity fluctuations not only approach a limiting characteristic distribution governed by K, but the intensity also ceases to grow with increasing amplifier length. The characteristic fluctuations are the same as in the lossless amplifier case.

Differences clearly exist in the counterpropagating case. This result was observed earlier for two beams.[25] The form in Eq. (6.7) can be easily generalized for any number of beams traveling in each direction. Comparisons between the loss and lossless amplifiers in the two- and four-beam cases are shown in Table I. The differences in the correlation between copropagating and counterpropagating beams in the four-beam case are also displayed.

Clearly loss tends to slightly reduce the coupling between the counterpropagating signals. We showed in the earlier work[25] that there are two distinct processes at work. The simple gain saturation reduces the fluctuations to the lossless case. When the loss term becomes important there is a second reduction of the fluctuations due to the loss decoupling. The separation of these two effects depends on the magnitude of K.

C. Saturating source

A measure of the effects of saturating the source term can be achieved by replacing $x_{\nu i}$ by

$$x_{\nu i}\left[1+\sum_{\lambda}x_\lambda(z_\lambda)\right]^{-1}.$$

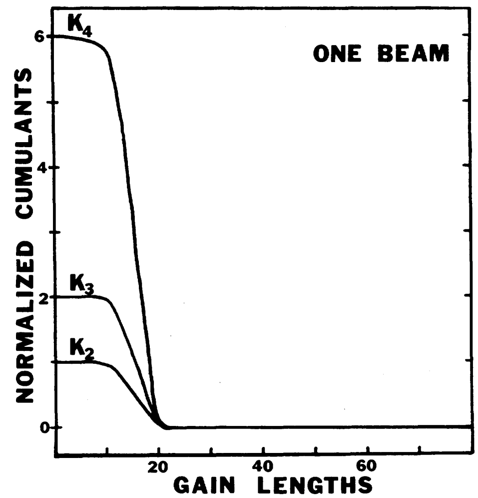

FIG. 1. Normalized cumulants versus gain lengths for a one-beam (no-loss) amplifier with input intensity 10^{-7}.

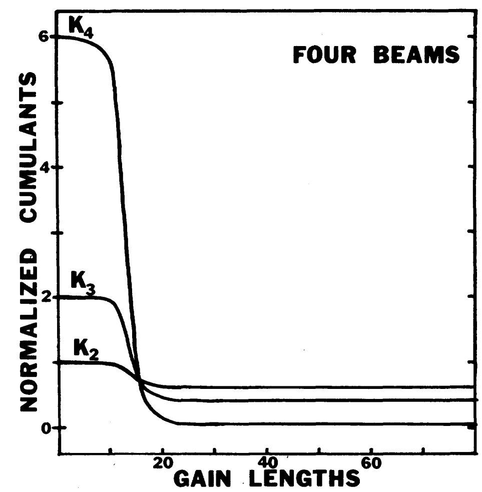

FIG. 3. Normalized cumulants versus gain lengths for four equal input beams (10^{-7}) in a no-loss amplifier.

That is, saturating the source by the sum of the intensities of all the beams evaluated at the end of the amplifier at which that source is applied. The results for two beams in a lossless amplifier are shown as an example in Table I. The effect of the source saturation is to heighten the competition between the beams leading to larger normalized variances in the fluctuations, and leading to greater overall fluctuations.

VII. RESULTS FOR VARIATION OF AMPLIFIER LENGTH

As an example of the length dependence of these results the case of the lossless amplifier has been solved for equal gain, equal source input, and one, two, and four beams. The results for various statistical quantities are shown in Figs. 1—3. Deviations from the initial thermal values occur as the combined intensities reach saturation levels (of order unity).

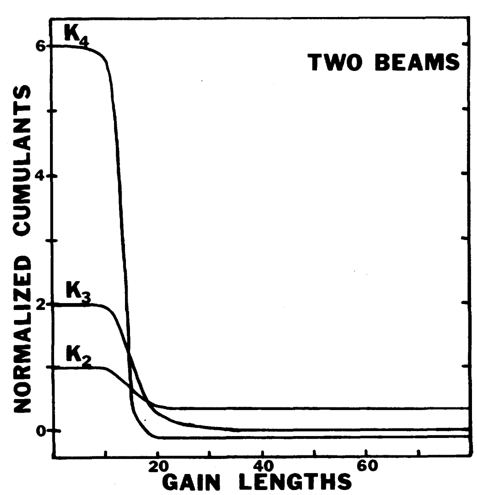

FIG. 2. Normalized cumulants versus gain lengths for two equal input (10^{-7}) beams in a no-loss amplifier.

VIII. CONCLUSIONS

These results indicate that the fluctuations of a single beam in a laser amplifier depend most significantly on the number of competing modes present. These effects will have to be taken into account in evaluating laser-amplifier performance, as not only will the spontaneous-emission noise be an additive contribution to the field of an amplified input signal, but the intensity-saturation coupling will cause additional noise.

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